

Fitting 3D Data with a Torus

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1 Introduction

The definition of a torus and some visualizations are found at the Wikipedia page [Torus](#). The canonical torus has center $\mathbf{C} = (0, 0, 0)$ and a normal line $\mathbf{N} = (0, 0, 1)$. The torus is generated by the revolution of a circle in the x_0x_2 -plane about the normal line. This document is concerned only about the case when the circle has x_0x_2 -center $(r_0, 0)$ and radius r_1 with $r_0 \geq r_1$. The projection of the torus onto the x_0x_1 -plane is an annulus whose x_0x_1 -center is $(0, 0)$, whose inner radius is $r_0 - r_1$ and whose outer radius is $r_0 + r_1$. A quartic polynomial equation that implicitly defines the torus is

$$(x_0^2 + x_1^2 + x_2^2 + r_0^2 - r_1^2)^2 - 4r_0^2(x_0^2 + x_1^2) = 0 \quad (1)$$

A parametric form for the torus point $\mathbf{X} = (x_0, x_1, x_2)$ is

$$\mathbf{X}(\alpha, \beta) = ((r_0 + r_1 \cos(\beta)) \cos(\alpha), (r_0 + r_1 \cos(\beta)) \sin(\alpha), r_1 \sin(\beta)) \quad (2)$$

for $\alpha \in [0, 2\pi)$ and $\beta \in [0, 2\pi)$.

Generally, let the torus center be \mathbf{C} with plane of symmetry containing \mathbf{C} and having mutually perpendicular and unit-length directions \mathbf{D}_0 and \mathbf{D}_1 . The axis of symmetry is the line containing \mathbf{C} and having direction \mathbf{N} (the plane normal). The distance from the center of the torus to the center of the tube is r_0 and the radius of the tube of the torus is r_1 . A point \mathbf{X} may be written as

$$\mathbf{X} = \mathbf{C} + y_0\mathbf{D}_0 + y_1\mathbf{D}_1 + y_2\mathbf{N} \quad (3)$$

where $\{\mathbf{D}_0, \mathbf{D}_1, \mathbf{N}\}$ is a right-handed orthonormal set. Therefore, $y_0 = \mathbf{D}_0 \cdot (\mathbf{P} - \mathbf{C})$, $y_1 = \mathbf{D}_1 \cdot (\mathbf{P} - \mathbf{C})$ and $y_2 = \mathbf{N} \cdot (\mathbf{P} - \mathbf{C})$. The implicit form of the torus is

$$F(\mathbf{X}; \mathbf{C}, \mathbf{N}, r_0^2, r_1^2) = (|\mathbf{X} - \mathbf{C}|^2 + r_0^2 - r_1^2)^2 - 4r_0^2(|\mathbf{X} - \mathbf{C}|^2 - (\mathbf{N} \cdot (\mathbf{X} - \mathbf{C}))^2) = 0 \quad (4)$$

The notation of F indicates that \mathbf{X} is the variable and the remaining parameters are considered constants. Observe that \mathbf{D}_0 and \mathbf{D}_1 are not present in the equation, which is to be expected by the symmetry. The parametric form is

$$\mathbf{X}(\alpha, \beta) = \mathbf{C} + (r_0 + r_1 \cos(\beta))(\cos(\alpha)\mathbf{D}_0 + \sin(\alpha)\mathbf{D}_1) + r_1 \sin(\beta)\mathbf{N} \quad (5)$$

for $\alpha \in [0, 2\pi)$ and $\beta \in [0, 2\pi)$.

2 Fitting Using a Least-Squares Approach

Let the sample points be $\{\mathbf{X}_i\}_{i=0}^{n-1}$. The center is $\mathbf{C} = (c_0, c_1, c_2)$. The normal vector is unit length, so $\mathbf{N} = (\cos \theta \sin \phi, \sin \theta \sin \phi, \cos \phi)$ for $\theta \in [0, 2\pi)$ and $\phi \in [0, \pi]$. Define $u = r_0^2$ and $v = r_0^2 - r_1^2$ with $u \geq v > 0$. Define the least-squares error function

$$E(\mathbf{p}) = E(c_0, c_1, c_2, \theta, \phi, u, v) = \sum_{i=0}^{n-1} [F(\mathbf{X}_i; \mathbf{C}, \mathbf{N}, r_0^2, r_1^2)]^2 \quad (6)$$

which depends on 7 parameters $\mathbf{p} = (c_0, c_1, c_2, \theta, \phi, u, v)$. Any of the standard minimization techniques for nonlinear least squares can be used to estimate the parameters. Implementations using the Gauss-Newton method and Levenberg-Marquardt method are found in [GteApprTorus3.h](#).

When the sample points are distributed so that there is large coverage by a purported fitted torus, a variation on fitting is the following. Compute the least-squares plane with origin \mathbf{C} and normal \mathbf{N} that fits the points. Define $G(\mathbf{X}; u, v) = F(\mathbf{X}; \mathbf{C}, \mathbf{N}, u, v)$. The only variables now are u and v . Define $L_i = |\mathbf{X}_i - \mathbf{C}|^2$ and $S_i = 4(L_i - \mathbf{N} \cdot (\mathbf{X}_i - \mathbf{C}))^2$. Define the error function

$$H(u, v) = \sum_{i=0}^{n-1} G(\mathbf{X}_i; u, v)^2 = \sum_{i=0}^{n-1} ((v + L_i)^2 - S_i u)^2 \quad (7)$$

The first-order partial derivatives are

$$\frac{dH}{du} = -2 \sum_{i=0}^{n-1} ((v + L_i)^2 - S_i u) S_i, \quad \frac{dH}{dv} = 4 \sum_{i=0}^{n-1} ((v + L_i)^2 - S_i u) (v + L_i) \quad (8)$$

Setting these to zero and expanding the terms, we have

$$0 = a_2 v^2 + a_1 v + a_0 - b_0 u, \quad 0 = c_3 v^3 + c_2 v^2 + c_1 v + c_0 - u(d_1 v + d_0) \quad (9)$$

where

$$a_2 = \sum_{i=0}^{n-1} S_i, \quad a_1 = 2 \sum_{i=0}^{n-1} S_i L_i, \quad a_0 = \sum_{i=0}^{n-1} S_i L_i^2, \quad b_0 = \sum_{i=0}^{n-1} S_i^2 \quad (10)$$

and

$$c_3 = \sum_{i=0}^{n-1} 1 = n, \quad c_2 = 3 \sum_{i=0}^{n-1} L_i, \quad c_1 = 3 \sum_{i=0}^{n-1} L_i^2, \quad c_0 = \sum_{i=0}^{n-1} L_i^3, \quad d_1 = \sum_{i=0}^{n-1} S_i = a_2, \quad d_0 = \sum_{i=0}^{n-1} S_i L_i = a_1/2 \quad (11)$$

The first equation of (9) is solved for

$$u = (a_2 v^2 + a_1 v + a_0)/b_0 = e_2 v^2 + e_1 v + e_0 \quad (12)$$

where the second equality defines the coefficients e_0 , e_1 and e_2 . Substitute this into the second equation of (9) to obtain a cubic polynomial equation

$$0 = f_3 v^3 + f_2 v^2 + f_1 v + f_0 \quad (13)$$

where $f_3 = c_3 - d_1 e_2$, $f_2 = c_2 - d_1 e_1 - d_0 e_2$, $f_1 = c_1 - d_1 e_0 - d_0 e_1$ and $f_0 = c_0 - d_0 e_0$. The positive v -roots are computed. For each root compute the corresponding u . For all pairs (u, v) with $u > v > 0$, evaluate $H(u, v)$ and choose the pair that minimizes $H(u, v)$. The torus radii are $r_0 = \sqrt{u}$ and $r_1 = \sqrt{u - v}$.

An implementation of this algorithm is found in [GteApprTorus3.h](#).