# Fitting 3D Data with a Torus

David Eberly, Geometric Tools, Redmond WA 98052
https://www.geometrictools.com/



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# 1 Introduction

The definition of a torus and some visualizations are found at the Wikipedia page Torus. The canonical torus has center $\mathbf{C} = (0,0,0)$ and a normal line $\mathbf{N} = (0,0,1)$. The torus is generated by the revolution of a circle in the $x_0x_2$-plane about the normal line. This document is concerned only about the case when the circle has $x_0x_2$-center $(r_0, 0)$ and radius $r_1$ with $r_0 \geq r_1$. The projection of the torus onto the $x_0x_1$-plane is an annulus whose $x_0x_1$-center is $(0,0)$, whose inner radius is $r_0 - r_1$ and whose outer radius is $r_0 + r_1$. A quartic polynomial equation that implicitly defines the torus is

$$\left(x_0^2 + x_1^2 + x_2^2 + r_0^2 - r_1^2\right)^2 - 4r_0^2\left(x_0^2 + x_1^2\right) = 0 \tag{1}$$

A parametric form for the torus point $\mathbf{X} = (x_0, x_1, x_2)$ is

$$\mathbf{X}(\alpha, \beta) = ((r_0 + r_1\cos(\beta))\cos(\alpha), (r_0 + r_1\cos(\beta))\sin(\alpha), r_1\sin(\beta)) \tag{2}$$

for $\alpha \in [0, 2\pi)$ and $\beta \in [0, 2\pi)$.

Generally, let the torus center be $\mathbf{C}$ with plane of symmetry containing $\mathbf{C}$ and having mutually perpendicular and unit-length directions $\mathbf{D}_0$ and $\mathbf{D}_1$. The axis of symmetry is the line containing $\mathbf{C}$ and having direction $\mathbf{N}$ (the plane normal). The distance from the center of the torus to the center of the tube is $r_0$ and the radius of the tube of the torus is $r_1$. A point $\mathbf{X}$ may be written as

$$\mathbf{X} = \mathbf{C} + y_0\mathbf{D}_0 + y_1\mathbf{D}_1 + y_2\mathbf{N} \tag{3}$$

where $\{\mathbf{D}_0, \mathbf{D}_1, \mathbf{N}\}$ is a right-handed orthonormal set. Therefore, $y_0 = \mathbf{D}_0 \cdot (\mathbf{P} - \mathbf{C})$, $y_1 = \mathbf{D}_1 \cdot (\mathbf{P} - \mathbf{C})$ and $y_2 = \mathbf{N} \cdot (\mathbf{P} - \mathbf{C})$. The implicit form of the torus is

$$F(\mathbf{X}; \mathbf{C}, \mathbf{N}, r_0^2, r_1^2) = \left(|\mathbf{X} - \mathbf{C}|^2 + r_0^2 - r_1^2\right)^2 - 4r_0^2\left(|\mathbf{X} - \mathbf{C}|^2 - (\mathbf{N} \cdot (\mathbf{X} - \mathbf{C}))^2\right) = 0 \tag{4}$$

The notation or $F$ indicates that $\mathbf{X}$ is the variable and the remaining parameters are considered constants. Observe that $\mathbf{D}_0$ and $\mathbf{D}_1$ are not present in the equation, which is to be expected by the symmetry. The parametric form is

$$\mathbf{X}(\alpha, \beta) = \mathbf{C} + (r_0 + r_1\cos(\beta))(\cos(\alpha)\mathbf{D}_0 + \sin(\alpha)\mathbf{D}_1) + r_1\sin(\beta)\mathbf{N} \tag{5}$$

for $\alpha \in [0, 2\pi)$ and $\beta \in [0, 2\pi)$.

# 2 Fitting Using a Least-Squares Approach

Let the sample points be $\{\mathbf{X}_i\}_{i=0}^{n-1}$. The center is $\mathbf{C} = (c_0, c_1, c_2)$. The normal vector is unit length, so $\mathbf{N} = (\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi)$ for $\theta \in [0, 2\pi)$ and $\phi \in [0, \pi]$. Define $u = r_0^2$ and $v = r_0^2 - r_1^2$ with $u \geq v > 0$. Define the least-squares error function

$$E(\mathbf{p}) = E(c_0, c_1, c_2, \theta, \phi, u, v) = \sum_{i=0}^{n-1}\left[F(\mathbf{X}_i; \mathbf{C}, \mathbf{N}, r_0^2, r_1^2)\right]^2 \tag{6}$$

which depends on 7 parameters $\mathbf{p} = (c_0, c_1, c_2, \theta, \phi, u, v)$. Any of the standard minimization techniques for nonlinear least squares can be used to estimate the parameters. Implementations using the Gauss–Newton method and Levenberg–Marquardt method are found in ApprTorus3.h.

When the sample points are distributed so that there is large coverage by a purported fitted torus, a variation on fitting is the following. Compute the least-squares plane with origin $\mathbf{C}$ and normal $\mathbf{N}$ that fits the points. Define $G(\mathbf{X}; u, v) = F(\mathbf{X}; \mathbf{C}, \mathbf{N}, u, v)$. The only variables now are $u$ and $v$. Define $L_i = |\mathbf{X}_i - \mathbf{C}|^2$ and $S_i = 4(L_i - \mathbf{N} \cdot (\mathbf{X}_i - \mathbf{C}))^2)$. Define the error function

$$H(u, v) = \sum_{i=0}^{n-1} G(\mathbf{X}_i; u, v)^2 = \sum_{i=0}^{n-1} ((v + L_i)^2 - S_i u)^2 \tag{7}$$

The first-order partial derivatives are

$$\frac{dH}{du} = -2 \sum_{i=0}^{n-1} ((v + L_i)^2 - S_i u) S_i, \quad \frac{dH}{dv} = 4 \sum_{i=0}^{n-1} ((v + L_i)^2 - S_i u)(v + L_i) \tag{8}$$

Setting these to zero and expanding the terms, we have

$$0 = a_2 v^2 + a_1 v + a_0 - b_0 u, \quad 0 = c_3 v^3 + c_2 v^2 + c_1 v + c0 - u(d_1 v + d_0) \tag{9}$$

where

$$a_2 = \sum_{i=0}^{n-1} S_i, \quad a_1 = 2 \sum_{i=0}^{n-1} S_i L_i, \quad a_2 = \sum_{i=0}^{n-1} S_i L_i^2, \quad b_0 = \sum_{i=0}^{n-1} S_i^2 \tag{10}$$

and

$$c_3 = \sum_{i=0}^{n-1} 1 = n, \quad c_2 = 3 \sum_{i=0}^{n-1} L_i, \quad c_1 = 3 \sum_{i=0}^{n-1} L_i, \quad c_0 = \sum_{i=0}^{n-1} L_i^3, \quad d_1 = \sum_{i-0}^{n-1} S_i = a_2, \quad d_0 = \sum_{i-0}^{n-1} S_i L_i = a_1/2 \tag{11}$$

The first equation of (9) is solved for

$$u = (a_2 v^2 + a_1 v + a_0)/b_0 = e_2 v^2 + e_1 v + e_0 \tag{12}$$

where the second equality defines the coefficients $e_0$, $e_1$ and $e_2$. Substitute this into the second equation of (9) to obtain a cubic polynomial equation

$$0 = f_3 v^3 + f_2 v^2 + f_1 v + f_0 \tag{13}$$

where $f_3 = c_3 - d_1 e_2$, $f_2 = c_2 - d_1 e_1 - d_0 e_2$, $f_1 = c_1 - d_1 e_0 - d_0 e_1$ and $f_0 = c_0 - d_0 e_0$. The positive $v$-roots are computed. For each root compute the corresponding $u$. For all pairs $(u, v)$ with $u > v > 0$, evaluate $H(u, v)$ and choose the pair that minimizes $H(u, v)$. The torus radii are $r_0 = \sqrt{u}$ and $r_1 = \sqrt{u - v}$.

An implementation of this algorithm is found in ApprTorus3.h.