Computing a Point of Reflection on a Sphere

David Eberly, Geometric Tools, Redmond WA 98052
https://www.geometrictools.com/

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1 Introduction

A point light is located at position $L$, which is more than one unit of distance from the origin. A sphere of radius 1 and centered at the origin will reflect light rays from the point light. A point $S$ outside the sphere potentially receives a reflected ray of light. If it does, we wish to compute the point $N$ on the sphere at which the light ray is reflected to reach the point $S$.

Imagine the smallest-angled single-sided cone whose vertex is $L$ and that contains the sphere. Figure 1 illustrates this.

**Figure 1.** The smallest-angled single-sided cone for the light position $L$ and the sphere. The point $S$ receives light as long as it is outside the sphere and not inside the shadowed area hidden from the light position. The point $L'$ is the reflection of $L$ through the ray whose origin is the sphere center and whose direction is $N$.

The point $S$ will receive a reflected light ray as long as it is outside the sphere and not in the shadow of the sphere, as illustrated in Figure 1. The point $L'$ is the reflection of $L$ through the ray whose origin is the sphere center and whose direction is $N$.

The key observation is that $N$ must be chosen so that the vectors $S - N$ and $L' - N$ are parallel and in the same direction.

2 Reflection of a Vector

Figure 1 illustrates the reflection $L'$ of the point $L$ through the normal ray. We need to compute this reflection. Figure 2 helps to understand how to do this.
Figure 2. The reflection $L'$ of $L$ through a ray with direction $N$.

The point $L$ has a component in the $N$ direction. After projecting out that component, the remainder is in some direction $N^\perp$ that is perpendicular to $N$. Thus, we may write

$$L = sN + tN^\perp$$  \hspace{1cm} (1)

where $s = N \cdot L$. The reflection through $N$ amounts to changing sign on the perpendicular component,

$$L' = sN - tN^\perp$$  \hspace{1cm} (2)

The sum of the vectors in Equations (1) and (2) is

$$L + L' = 2sN = 2(N \cdot L)N$$  \hspace{1cm} (3)

We may solve for the reflected vector,

$$L' = 2(N \cdot L)N - L$$  \hspace{1cm} (4)

3 Computing a Reflection Point

The assumption in this section is that $S$ and $L$ are not parallel vectors. For if they were, the point of reflection is $N = L/|L|$: that is, the light is reflected in the opposite direction to reach $S$. Our assumption has the consequence that $S \times L \neq 0$ (parallel vectors have a nonzero cross product).

Figure 1 shows that $N$ must bisect the rays whose common origin is $N$ and whose directions are $S - N$ and $L - N$. Therefore, we may represent

$$N = xS + yL$$  \hspace{1cm} (5)

for some scalars $x > 0$ and $y > 0$. Observe that

$$S \times N = yS \times L, \quad N \times L = xS \times L, \quad N \cdot L = xS \cdot L + yL \cdot L$$  \hspace{1cm} (6)

Because $N$ is a unit-length vector, we also know that

$$1 = N \cdot N = x^2S \cdot S + 2xyS \cdot L + y^2L \cdot L$$  \hspace{1cm} (7)
As noted in the introduction, \( \mathbf{N} \) must be chosen so that the vectors \( \mathbf{S} - \mathbf{N} \) and \( \mathbf{L}' - \mathbf{N} \) are parallel. Their cross product must be the zero vector,

\[
\mathbf{0} = (\mathbf{S} - \mathbf{N}) \times (\mathbf{L}' - \mathbf{N}) = (\mathbf{S} - \mathbf{N}) \times [(2\mathbf{N} \cdot \mathbf{L} - 1)\mathbf{N} - \mathbf{L}] = (2\mathbf{N} \cdot \mathbf{L} - 1)\mathbf{S} \times \mathbf{N} - \mathbf{S} \times \mathbf{L} + \mathbf{N} \times \mathbf{L} = [(2\mathbf{N} \cdot \mathbf{L} - 1)y - 1 + x] \mathbf{S} \times \mathbf{L} = [(2x\mathbf{S} \cdot \mathbf{L} + 2y\mathbf{L} \cdot \mathbf{L} - 1)y - 1 + x] \mathbf{S} \times \mathbf{L}
\]

where we have used Equations (2), (5), and (6). By assumption, \( \mathbf{S} \times \mathbf{L} \neq \mathbf{0} \), so Equation (8) implies

\[
(2x\mathbf{S} \cdot \mathbf{L} + 2y\mathbf{L} \cdot \mathbf{L} - 1)y - 1 + x = 0
\]

Equations (7) and (9) are two quadratic equations in two unknowns \( x \) and \( y \),

\[
p(x, y) = ax^2 + 2bxy + cy^2 - 1 = 0, \quad q(x, y) = 2bxy + 2cy^2 + x - y - 1 = 0
\]

where \( a = \mathbf{S} \cdot \mathbf{S}, \ b = \mathbf{S} \cdot \mathbf{L}, \) and \( c = \mathbf{L} \cdot \mathbf{L} \). We may solve \( q(x, y) = 0 \) for \( x \) in terms of \( y \),

\[
x = \frac{-2cy^2 + y + 1}{2by + 1}
\]

Substituting into the equation \( p(x, y) = 0 \), we have

\[
a(-2cy^2 + y + 1)^2 + 2by(2by + 1)(-2cy^2 + y + 1) + (cy^2 - 1)(2by + 1)^2 = 0
\]

The numerator of Equation (12) is the quartic polynomial,

\[
r(y) = 4c(ac - b^2)y^4 - 4(ac - b^2)y^3 + (a + 2b + c - 4ac)y^2 + 2(a - b)y + a - 1 = 0
\]

Observe that

\[
ac - b^2 = (\mathbf{S} \cdot \mathbf{S})(\mathbf{L} \cdot \mathbf{L}) - (\mathbf{S} \cdot \mathbf{L})^2 = |\mathbf{S} \times \mathbf{L}|^2 \neq 0
\]

so the coefficient of \( y^4 \) is not zero, which means \( r(y) \) really has degree 4.

Now compute the real-valued roots of \( r(y) = 0 \). For each root \( \bar{y} > 0 \), compute \( \bar{x} = (-2cy^2 + \bar{y} + 1)/(2b\bar{y} + 1) \) from Equation (11). Of all the pairs \((\bar{x}, \bar{y})\), select that pair for which \( \bar{x} > 0 \) and \( \bar{y} > 0 \). The point of reflection is \( \mathbf{N} = \bar{x}\mathbf{S} + \bar{y}\mathbf{L} \).