Rotation Representations

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Contents

This document is a summary of representations of rotations by matrices, quaternions, or axis-angle pairs. Conversions between the representations is provided. Interpolation methods for quaternions and for rotation matrices are discussed.

1 Matrix Representation

A 2D rotation is a tranformation of the form

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \tag{1}$$

where θ is the angle of rotation. A 3D rotation is a 2D rotation that is applied within a specified plane that contains the origin. Such a rotation can be represented by a 3×3 *rotation matrix* $R = [\mathbf{R}_0\ \mathbf{R}_1\ \mathbf{R}_2]$ whose columns $\mathbf{R}_0$, $\mathbf{R}_1$ and $\mathbf{R}_2$ form a right-handed orthonormal set. That is, $|\mathbf{R}_0| = |\mathbf{R}_1| = |\mathbf{R}_2| = 1$, $\mathbf{R}_0 \cdot \mathbf{R}_1 = \mathbf{R}_0 \cdot \mathbf{R}_2 = \mathbf{R}_1 \cdot \mathbf{R}_2 = 0$ and $\mathbf{R}_0 \cdot \mathbf{R}_1 \times \mathbf{R}_2 = 1$. The columns of the matrix correspond to the final rotated values of the standard basis vectors $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$, in that order. Given a 3×1 vector $\mathbf{X} = [x_j]$ and 3×3 rotation matrix $R = [r_{ij}]$, the rotated vector is

$$R\mathbf{X} = \left[\sum_{j=0}^{2} r_{ij} x_j \right] \tag{2}$$

2 Axis-Angle Representation

If the plane of rotation has unit length normal $\mathbf{W}$, then the *axis-angle representation* of the rotation is the pair $\langle \mathbf{W}, \theta \rangle$. The direction of rotation is chosen so that as you look down on the plane from the side to which $\mathbf{W}$ points, the rotation is counterclockwise about the origin for $\theta > 0$. This is the same convention used for a 2D rotation.

2.1 Axis-Angle to Matrix

If $\mathbf{U}$, $\mathbf{V}$, and $\mathbf{W}$ form a right-handed orthonormal set, then any point can be represented as $\mathbf{X} = u_0\mathbf{U} + v_0\mathbf{V} + w_0\mathbf{W}$. Rotation of $\mathbf{X}$ about the axis $\mathbf{W}$ by the angle θ produces $R\mathbf{X} = u_1\mathbf{U} + v_1\mathbf{V} + w_1\mathbf{W}$. Clearly from the geometry, $w_1 = w_0 = \mathbf{W} \cdot \mathbf{X}$. The other two components are changed as if a 2D rotation has been applied to them, so $u_1 = \cos(\theta)u_0 - \sin(\theta)v_0$ and $v_1 = \sin(\theta)u_0 + \cos(\theta)v_0$. Using the right-handedness of the orthonormal set, it is easily shown that

$$\mathbf{W} \times \mathbf{X} = u_0\mathbf{W} \times \mathbf{U} + v_0\mathbf{W} \times \mathbf{V} + w_0\mathbf{W} \times \mathbf{W} = -v_0\mathbf{U} + u_0\mathbf{V} \tag{3}$$

and

$$\mathbf{W} \times (\mathbf{W} \times \mathbf{X}) = -v_0\mathbf{W} \times \mathbf{U} + u_0\mathbf{W} \times \mathbf{V} = -u_0\mathbf{U} - v_0\mathbf{V} \tag{4}$$

Combining these in the form shown and using the relationship between u_0, v_0, u_1 and v_1 produces

$$\begin{aligned}(\sin\theta)\mathbf{W}\times\mathbf{X} + (1-\cos\theta)\mathbf{W}\times(\mathbf{W}\times\mathbf{X}) &= (-v_0\sin\theta - u_0(1-\cos\theta))\mathbf{U} + (u_0\sin\theta - v_0(1-\cos\theta))\mathbf{V} \\ &= (u_1-u_0)\mathbf{U} + (v_1-v_0)\mathbf{V} \\ &= R\mathbf{X}-\mathbf{X}\end{aligned} \tag{5}$$

Therefore, the rotation of $\mathbf{X}$ given the axis $\mathbf{W}$ and angle θ is

$$R\mathbf{X} = \mathbf{X} + (\sin\theta)\mathbf{W}\times\mathbf{X} + (1-\cos\theta)\mathbf{W}\times(\mathbf{W}\times\mathbf{X}) \tag{6}$$

This can also be written in matrix form by defining the following where $\mathbf{W} = (a, b, c)$,

$$S = \begin{bmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{bmatrix} \tag{7}$$

in which case

$$R = I + (\sin\theta)S + (1-\cos\theta)S^2 \tag{8}$$

and consequently $R\mathbf{X} = \mathbf{X} + (\sin\theta)S\mathbf{X} + (1-\cos\theta)S^2\mathbf{X}$.

2.2 Matrix to Axis-Angle

The inverse problem is to start with the rotation matrix and extract an angle and unit-length axis. There are multiple solutions because $-\mathbf{W}$ is a valid axis whenever $\mathbf{W}$ is and $\theta + 2\pi k$ is a valid solution whenever θ is. First, the *trace* of a matrix is defined to be the sum of the diagonal terms. Some algebra will show that $\cos\theta = (\mathrm{Trace}(R) - 1)/2$, in which case

$$\theta = \cos^{-1}((\mathrm{Trace}(R)-1)/2) \in [0, \pi]. \tag{9}$$

Also, it is easily shown that

$$R - R^{\mathsf{T}} = (2\sin\theta)S \tag{10}$$

where S is a skew-symmetric matrix. The constructions below are based on the cases $\theta = 0$, $\theta \in (0, \pi)$ and $\theta = \pi$.

If $\theta = 0$, then any unit-length diretion vector for the axis is valid because there is no rotation.

If $\theta \in (0, \pi)$, equation (10) allows direct extraction of the axis, $\mathbf{D} = (r_{21} - r_{12}, r_{02} - r_{20}, r_{10} - r_{01})$ and $\mathbf{W} = \mathbf{D}/|\mathbf{D}|$.

If $\theta = \pi$, equation (10) does not help with constructing the axis because $R - R^{\mathsf{T}} = 0$. In this case note that

$$R = I + 2S^2 = \begin{bmatrix} 1-2(w_1^2+w_2^2) & 2w_0w_1 & 2w_0w_2 \\ 2w_0w_1 & 1-2(w_0^2+w_2^2) & 2w_1w_2 \\ 2w_0w_2 & 2w_1w_2 & 1-2(w_0^2+w_1^2) \end{bmatrix} \tag{11}$$

where $\mathbf{W} = (w_0, w_1, w_2)$. The idea is to extract the maximum component of the axis from the diagonal entries of the rotation matrix. If r_{00} is maximum, then w_0 must be the largest component in magnitude. Compute $4w_0^2 = r_{00} - r_{11} - r_{22} + 1$ and select $w_0 = \sqrt{r_{00} - r_{11} - r_{22} + 1}/2$. Consequently, $w_1 = r_{01}/(2w_0)$ and $w_2 = r_{02}/(2w_0)$. If r_{11} is maximum, then compute $4w_1^2 = r_{11} - r_{00} - r_{22} + 1$ and select $w_1 = \sqrt{r_{11} - r_{00} - r_{22} + 1}/2$. Consequently, $w_0 = r_{01}/(2w_1)$ and $w_2 = r_{12}/(2w_1)$. Finally, if r_{22} is maximum, then compute $4w_2^2 = r_{22} - r_{00} - r_{11} + 1$ and select $w_2 = \sqrt{r_{22} - r_{00} - r_{11} + 1}/2$. Consequently, $w_0 = r_{02}/(2w_2)$ and $w_1 = r_{12}/(2w_2)$.

3 Quaternion Representation

A third representation involves *unit quaternions*. Only a summary is provided here. A unit quaternion is denoted by $q = w + xi + yj + zk$ where w, x, y and z are real numbers and where the 4-tuple (w, x, y, z) is unit length. The set of unit quaternions is the unit hypersphere in $\mathbb{R}^4$. The products of i, j and k are defined by $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$, $jk = -kj = i$ and $ki = -ik = j$. Observe that the products are *not commutative*. The product of two unit quaternions $q_n = w_n + x_n i + y_n j + z_n k$ for $n = 0, 1$ is defined by distributing the product over the sums, keeping in mind that the order of operands is important,

$$\begin{aligned} q_0 q_1 = \; & (w_0 w_1 - x_0 x_1 - y_0 y_1 - z_0 z_1) + \\ & (w_0 x_1 + x_0 w_1 + y_0 z_1 - z_0 y_1) i + \\ & (w_0 y_1 - x_0 z_1 + y_0 w_1 + z_0 x_1) j + \\ & (w_0 z_1 + x_0 y_1 - y_0 x_1 + z_0 w_1) k \end{aligned} \tag{12}$$

The conjugate of q is defined by

$$q^* = w - xi - yj - zk \tag{13}$$

Observe that $qq^* = q^*q = 1$ where the right-hand side 1 is the w-term of the quaternion, the x-, y- and z-terms being all 0.

3.1 Axis-Angle to Quaternion

If $\mathbf{A} = (x_0, y_0, z_0)$ is the unit length axis of rotation and if θ is the angle of rotation, a quaternion $q = w + xi + yj + zk$ that represents the rotation satisfies $w = \cos(\theta/2)$, $x = x_0 \sin(\theta/2)$, $y = y_0 \sin(\theta/2)$ and $z = z_0 \sin(\theta/2)$.

If a vector $\mathbf{V} = (v_0, v_1, v_2)$ is represented as the quaternion $\hat{v} = v_0 i + v_1 j + v_2 k$, and if q represents a rotation, then the rotated vector $\mathbf{U}$ is represented by quaternion $\hat{u} = u_0 i + u_1 j + u_2 k$ where

$$\hat{u} = q \hat{v} q^*. \tag{14}$$

It can be shown that the w-term of $\hat{u}$ must really be 0.

3.2 Quaternion to Axis-Angle

Let $q = w + xi + yj + zk$ be a unit quaternion. If $|w| = 1$, then the angle is $\theta = 0$ and any unit-length direction vector for the axis will do because there is no rotation. If $|w| < 1$, the angle is obtained as $\theta = 2\cos^{-1}(w)$ and the axis is computed as $\mathbf{A} = (x, y, z)/\sqrt{1 - w^2}$.

3.3 Quaternion to Matrix

Using the identities $2\sin^2(\theta/2) = 1 - \cos(\theta)$ and $\sin(\theta) = 2\sin(\theta/2)\cos(\theta/2)$, it is easily shown that $2wx = (\sin\theta)w_0$, $2wy = (\sin\theta)w_1$, $2wz = (\sin\theta)w_2$, $2x^2 = (1-\cos\theta)w_0^2$, $2xy = (1-\cos\theta)w_0w_1$, $2xz = (1-\cos\theta)w_0w_2$, $2y^2 = (1-\cos\theta)w_1^2$, $2yz = (1-\cos\theta)w_1w_2$ and $2z^2 = (1-\cos\theta)w_2^2$. The right-hand sides of all these equations are terms in the expression $R = I + (\sin\theta)S + (1-\cos\theta)S^2$. Replacing them yields

$$R = \begin{bmatrix} 1-2y^2-2z^2 & 2xy-2wz & 2xz+2wy \\ 2xy+2wz & 1-2x^2-2z^2 & 2yz-2wx \\ 2xz-2wy & 2yz+2wx & 1-2x^2-2y^2 \end{bmatrix} \tag{15}$$

3.4 Matrix to Quaternion

Previously it was mentioned that $\cos\theta = (\text{Trace}(R)-1)/2$. Using the identity $2\cos^2(\theta/2) = 1+\cos\theta$ yields $w^2 = \cos^2(\theta/2) = (\text{Trace}(R)+1)/4$ or $|w| = \sqrt{\text{Trace}(R)+1}/2$. If $\text{Trace}(R) > 0$, then $|w| > 1/2$, so without loss of generality choose w to be the positive square root, $w = \sqrt{\text{Trace}(R)+1}/2$. The identity $R - R^{\mathsf{T}} = (2\sin\theta)S$ also yielded $(r_{12}-r_{21}, r_{20}-r_{02}, r_{01}-r_{10}) = 2\sin\theta(w_0, w_1, w_2)$. Finally, identities derived earlier were $2xw = w_0\sin\theta$, $2yw = w_1\sin\theta$, and $2zw = w_2\sin\theta$. Combining these leads to $x = (r_{12}-r_{21})/(4w)$, $y = (r_{20}-r_{02})/(4w)$, and $z = (r_{01}-r_{10})/(4w)$.

If $\text{Trace}(R) \le 0$, then $|w| \le 1/2$. The idea is to extract first the largest one of x, y, or z from the diagonal terms of the rotation R in equation 15. If r_{00} is the maximum diagonal term, then x is larger in magnitude than y or z. Some algebra shows that $4x^2 = r_{00}-r_{11}-r_{22}+1$ from which is chosen $x = \sqrt{r_{00}-r_{11}-r_{22}+1}/2$. Consequently, $w = (r_{12}-r_{21})/(4x)$, $y = (r_{01}+r_{10})/(4x)$ and $z = (r_{02}+r_{20})/(4x)$. If r_{11} is the maximum diagonal term, then compute $4y^2 = r_{11}-r_{00}-r_{22}+1$ and choose $y = \sqrt{r_{11}-r_{00}-r_{22}+1}/2$. Consequently, $w = (r_{20}-r_{02})/(4y)$, $x = (r_{01}+r_{10})/(4y)$ and $z = (r_{12}+r_{21})/(4y)$. Finally, if r_{22} is the maximum diagonal term, then compute $4z^2 = r_{22}-r_{00}-r_{11}+1$ and choose $z = \sqrt{r_{22}-r_{00}-r_{11}+1}/2$. Consequently, $w = (r_{01}-r_{10})/(4z)$, $x = (r_{02}+r_{20})/(4z)$, and $y = (r_{12}+r_{21})/(4z)$.

4 Interpolation

4.1 Quaternion Interpolation

Quaternions are quite amenable to interpolation. The standard operation that is used is *spherical linear interpolation*, affectionately known as *slerp*. Given quaternions p and q with acute angle θ between them, slerp is defined as $s(t; p, q) = p(p^*q)^t$ for $t \in [0, 1]$. Note that $s(0; p, q) = p$ and $s(1; p, q) = q$. An equivalent definition of slerp that is more amenable to calculation is

$$s(t; p, q) = \frac{\sin((1-t)\theta)p + \sin(t\theta)q}{\sin(\theta)}. \tag{16}$$

If p and q are thought of as points on a unit circle, the formula above is a parameterization of the shortest arc between them. If a particle travels on that curve according to the parameterization, it does so with constant speed. Thus, any uniform sampling of t in $[0, 1]$ produces equally spaced points on the arc.

We assume that only p, q, and t are specified. Moreover, because q and $-q$ represent the same rotation, you can replace q by $-q$ if necessary to guarantee that the angle between p and q treated as 4-tuples is acute. That is, $p \cdot q \geq 0$. As 4-tuples, p and q are unit length. The dot product is therefore $p \cdot q = \cos(\theta)$.

4.2 Rotation Matrix Interpolation

The absence of a meaningful interpolation formula that directly applies to rotation matrices is used as an argument for the superiority of quaternions over rotation matrices. However, rotations can be interpolated directly in a way equivalent to what slerp produces. If P and Q are rotations corresponding to quaternions p and q, the slerp of the matrices is

$$S(t; P, Q) = P(P^{\mathsf{T}}Q)^t, \tag{17}$$

the same formula that defines slerp for quaternions. The technical problem is to define what is meant by R^t for a rotation R and real-valued t. If the rotation has axis $\mathbf{A}$ and angle θ, then R^t has the same rotation axis, but the angle of rotation is $t\theta$. The procedure for computing slerp of the rotation matrices is

1. Compute $R = P^{\mathsf{T}}Q$.
2. Extract an axis $\mathbf{A}$ and an angle θ from R.
3. Compute R^t by converting the axis-angle pair $\mathbf{A}$, $t\theta$.
4. Compute the $S(t; P, Q) = PR^t$.

4.3 Axis-Angle Interpolation

There is no obvious and natural way to produce the same interpolation that occurs with quaternions and rotation matrices. The only choice is to convert to one of the other representations, interpolate in that form, then convert the interpolated result back to axis-angle form.