

Quaternion Algebra and Calculus

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This document provides a mathematical summary of quaternion algebra and calculus and how they relate to rotations and interpolation of rotations. The ideas are based on the article [1].

1 Quaternion Algebra

A *quaternion* is given by $q = w + xi + yj + zk$ where $w, x, y,$ and z are real numbers. Define $q_n = w_n + x_n i + y_n j + z_n k$ ($n = 0, 1$). *Addition* and *subtraction* of quaternions is defined by

$$\begin{aligned} q_0 \pm q_1 &= (w_0 + x_0 i + y_0 j + z_0 k) \pm (w_1 + x_1 i + y_1 j + z_1 k) \\ &= (w_0 \pm w_1) + (x_0 \pm x_1) i + (y_0 \pm y_1) j + (z_0 \pm z_1) k. \end{aligned} \quad (1)$$

Multiplication for the primitive elements $i, j,$ and k is defined by $i^2 = j^2 = k^2 = -1, ij = -ji = k, jk = -kj = i,$ and $ki = -ik = j$. *Multiplication* of quaternions is defined by

$$\begin{aligned} q_0 q_1 &= (w_0 + x_0 i + y_0 j + z_0 k)(w_1 + x_1 i + y_1 j + z_1 k) \\ &= (w_0 w_1 - x_0 x_1 - y_0 y_1 - z_0 z_1) + \\ &\quad (w_0 x_1 + x_0 w_1 + y_0 z_1 - z_0 y_1) i + \\ &\quad (w_0 y_1 - x_0 z_1 + y_0 w_1 + z_0 x_1) j + \\ &\quad (w_0 z_1 + x_0 y_1 - y_0 x_1 + z_0 w_1) k. \end{aligned} \quad (2)$$

Multiplication is not commutative in that the products $q_0 q_1$ and $q_1 q_0$ are not necessarily equal.

The *conjugate* of a quaternion is defined by

$$q^* = (w + xi + yj + zk)^* = w - xi - yj - zk. \quad (3)$$

The conjugate of a product of quaternions satisfies the properties $(p^*)^* = p$ and $(pq)^* = q^* p^*$.

The *norm* of a quaternion is defined by

$$N(q) = N(w + xi + yj + zk) = w^2 + x^2 + y^2 + z^2. \quad (4)$$

The norm is a real-valued function and the norm of a product of quaternions satisfies the properties $N(q^*) = N(q)$ and $N(pq) = N(p)N(q)$.

The *multiplicative inverse* of a quaternion q is denoted q^{-1} and has the property $qq^{-1} = q^{-1}q = 1$. It is constructed as

$$q^{-1} = q^*/N(q) \quad (5)$$

where the division of a quaternion by a real-valued scalar is just componentwise division. The inverse operation satisfies the properties $(p^{-1})^{-1} = p$ and $(pq)^{-1} = q^{-1}p^{-1}$.

A simple but useful function is the *selection* function

$$W(q) = W(w + xi + yj + zk) = w \quad (6)$$

which selects the “real part” of the quaternion. This function satisfies the property $W(q) = (q + q^*)/2$.

The quaternion $q = w + xi + yj + zk$ may also be viewed as $q = w + \hat{v}$ where $\hat{v} = xi + yj + zk$. If we identify \hat{v} with the 3D vector (x, y, z) , then quaternion multiplication can be written using vector dot product (\bullet) and cross product (\times) as

$$(w_0 + \hat{v}_0)(w_1 + \hat{v}_1) = (w_0w_1 - \hat{v}_0 \bullet \hat{v}_1) + w_0\hat{v}_1 + w_1\hat{v}_0 + \hat{v}_0 \times \hat{v}_1. \quad (7)$$

In this form it is clear that $q_0q_1 = q_1q_0$ if and only if $\hat{v}_0 \times \hat{v}_1 = 0$ (these two vectors are parallel).

A quaternion q may also be viewed as a 4D vector (w, x, y, z) . The *dot product* of two quaternions is

$$q_0 \bullet q_1 = w_0w_1 + x_0x_1 + y_0y_1 + z_0z_1 = W(q_0q_1^*). \quad (8)$$

A *unit quaternion* is a quaternion q for which $N(q) = 1$. The inverse of a unit quaternion and the product of unit quaternions are themselves unit quaternions. A unit quaternion can be represented by

$$q = \cos \theta + \hat{u} \sin \theta \quad (9)$$

where \hat{u} as a 3D vector has length 1. However, observe that the quaternion product $\hat{u}\hat{u} = -1$. Note the similarity to unit length complex numbers $\cos \theta + i \sin \theta$. In fact, Euler's identity for complex numbers generalizes to quaternions,

$$\exp(\hat{u}\theta) = \cos \theta + \hat{u} \sin \theta, \quad (10)$$

where the exponential on the left-hand side is evaluated by symbolically substituting $\hat{u}\theta$ into the power series representation for $\exp(x)$ and replacing products $\hat{u}\hat{u}$ by -1 . From this identity it is possible to define the *power* of a unit quaternion,

$$q^t = (\cos \theta + \hat{u} \sin \theta)^t = \exp(\hat{u}t\theta) = \cos(t\theta) + \hat{u} \sin(t\theta). \quad (11)$$

It is also possible to define the *logarithm* of a unit quaternion,

$$\log(q) = \log(\cos \theta + \hat{u} \sin \theta) = \log(\exp(\hat{u}\theta)) = \hat{u}\theta. \quad (12)$$

It is important to note that the noncommutativity of quaternion multiplication disallows the standard identities for exponential and logarithm functions. The quaternions $\exp(p)\exp(q)$ and $\exp(p+q)$ are not necessarily equal. The quaternions $\log(pq)$ and $\log(p) + \log(q)$ are not necessarily equal.

2 Relationship of Quaternions to Rotations

A unit quaternion $q = \cos \theta + \hat{u} \sin \theta$ represents the rotation of the 3D vector \hat{v} by an angle 2θ about the 3D axis \hat{u} . The rotated vector, represented as a quaternion, is $R(\hat{v}) = q\hat{v}q^*$. The proof requires showing that $R(\hat{v})$ is a 3D vector, a length-preserving function of 3D vectors, a linear transformation, and does not have a reflection component.

To see that $R(\hat{v})$ is a 3D vector,

$$\begin{aligned}
W(R(\hat{v})) &= W(q\hat{v}q^*) \\
&= [(q\hat{v}q^*) + (q\hat{v}q^*)^*]/2 \\
&= [q\hat{v}q^* + q\hat{v}^*q^*]/2 \\
&= q[(\hat{v} + \hat{v}^*)/2]q^* \\
&= qW(\hat{v})q^* \\
&= W(\hat{v}) \\
&= 0.
\end{aligned}$$

To see that $R(\hat{v})$ is length-preserving,

$$\begin{aligned}
N(R(\hat{v})) &= N(q\hat{v}q^*) \\
&= N(q)N(\hat{v})N(q^*) \\
&= N(q)N(\hat{v})N(q) \\
&= N(\hat{v}).
\end{aligned}$$

To see that $R(\hat{v})$ is a linear transformation, let a be a real-valued scalar and let \hat{v} and \hat{w} be 3D vectors; then

$$\begin{aligned}
R(a\hat{v} + \hat{w}) &= q(a\hat{v} + \hat{w})q^* \\
&= (qa\hat{v}q^*) + (q\hat{w}q^*) \\
&= a(q\hat{v}q^*) + (q\hat{w}q^*) \\
&= aR(\hat{v}) + R(\hat{w}),
\end{aligned}$$

thereby showing that the transform of a linear combination of vectors is the linear combination of the transforms.

The previous three properties show that $R(\hat{v})$ is an orthonormal transformation. Such transformations include rotations *and* reflections. Consider R as a function of q for a *fixed vector* \hat{v} . That is, $R(q) = q\hat{v}q^*$. This function is a continuous function of q . For each q it is a linear transformation with determinant $D(q)$, so the determinant itself is a continuous function of q . Thus, $\lim_{q \rightarrow 1} R(q) = R(1) = I$, the identity function (the limit is taken along any path of quaternions which approach the quaternion 1) and $\lim_{q \rightarrow 1} D(q) = D(1) = 1$. By continuity, $D(q)$ is identically 1 and $R(q)$ does not have a reflection component.

Now we prove that the unit rotation axis is the 3D vector \hat{u} and the rotation angle is 2θ . To see that \hat{u} is a unit rotation axis we need only show that \hat{u} is unchanged by the rotation. Recall that $\hat{u}^2 = \hat{u}\hat{u} = -1$. This implies that $\hat{u}^3 = -\hat{u}$. Now

$$\begin{aligned}
R(\hat{u}) &= q\hat{u}q^* \\
&= (\cos \theta + \hat{u} \sin \theta)\hat{u}(\cos \theta - \hat{u} \sin \theta) \\
&= (\cos \theta)^2\hat{u} - (\sin \theta)^2\hat{u}^3 \\
&= (\cos \theta)^2\hat{u} - (\sin \theta)^2(-\hat{u}) \\
&= \hat{u}.
\end{aligned}$$

To see that the rotation angle is 2θ , let \hat{u} , \hat{v} , and \hat{w} be a right-handed set of orthonormal vectors. That is, the vectors are all unit length; $\hat{u} \bullet \hat{v} = \hat{u} \bullet \hat{w} = \hat{v} \bullet \hat{w} = 0$; and $\hat{u} \times \hat{v} = \hat{w}$, $\hat{v} \times \hat{w} = \hat{u}$, and $\hat{w} \times \hat{u} = \hat{v}$. The vector \hat{v} is rotated by an angle ϕ to the vector $q\hat{v}q^*$, so $\hat{v} \bullet (q\hat{v}q^*) = \cos(\phi)$. Using equation (8) and $\hat{v}^* = -\hat{v}$, and $\hat{p}^2 = -1$ for unit quaternions with zero real part,

$$\begin{aligned}
\cos(\phi) &= \hat{v} \bullet (q\hat{v}q^*) \\
&= W(\hat{v}^*q\hat{v}q^*) \\
&= W[-\hat{v}(\cos\theta + \hat{u}\sin\theta)\hat{v}(\cos\theta - \hat{u}\sin\theta)] \\
&= W[(-\hat{v}\cos\theta - \hat{v}\hat{u}\sin\theta)(\hat{v}\cos\theta - \hat{v}\hat{u}\sin\theta)] \\
&= W[-\hat{v}^2(\cos\theta)^2 + \hat{v}^2\hat{u}\sin\theta\cos\theta - \hat{v}\hat{u}\hat{v}\sin\theta\cos\theta + (\hat{v}\hat{u})^2(\sin\theta)^2] \\
&= W[(\cos\theta)^2 - (\sin\theta)^2 - (\hat{u} + \hat{v}\hat{u}\hat{v})\sin\theta\cos\theta]
\end{aligned}$$

Now $\hat{v}\hat{u} = -\hat{v} \bullet \hat{u} + \hat{v} \times \hat{u} = \hat{v} \times \hat{u} = -\hat{w}$ and $\hat{v}\hat{u}\hat{v} = -\hat{w}\hat{v} = \hat{w} \bullet \hat{v} - \hat{w} \times \hat{v} = \hat{u}$. Consequently,

$$\begin{aligned}
\cos(\phi) &= W[(\cos\theta)^2 - (\sin\theta)^2 - (\hat{u} + \hat{v}\hat{u}\hat{v})\sin\theta\cos\theta] \\
&= W[(\cos\theta)^2 - (\sin\theta)^2 - \hat{u}(2\sin\theta\cos\theta)] \\
&= (\cos\theta)^2 - (\sin\theta)^2 \\
&= \cos(2\theta)
\end{aligned}$$

and the rotation angle is $\phi = 2\theta$.

It is important to note that the quaternions q and $-q$ represent the same rotation since $(-q)\hat{v}(-q)^* = q\hat{v}q^*$. While either quaternion will do, the interpolation methods require choosing one over the other.

3 Quaternion Calculus

The only support we need for quaternion interpolation is to differentiate unit quaternion functions raised to a real-valued power. These formulas are identical to those derived in a standard calculus course, but the order of multiplication must be observed.

The derivative of the function q^t where q is a constant unit quaternion is

$$\frac{d}{dt}q^t = q^t \log(q) \tag{13}$$

where \log is the function defined earlier by $\log(\cos\theta + \hat{u}\sin\theta) = \hat{u}\theta$. To prove this, observe that

$$q^t = \cos(t\theta) + \hat{u}\sin(t\theta)$$

in which case

$$\frac{d}{dt}q^t = -\sin(t\theta)\theta + \hat{u}\cos(t\theta)\theta = \hat{u}\hat{u}\sin(t\theta)\theta + \hat{u}\cos(t\theta)\theta$$

where we have used $-1 = \hat{u}\hat{u}$. Factoring this, we have

$$\frac{d}{dt}q^t = (\hat{u}\sin(t\theta) + \cos(t\theta))\hat{u}\theta = q^t \log(q)$$

The right-hand side also factors as $\log(q)q^t$. Generally, the order of operands in a quaternion multiplication is important, but not in this special case. The power can be a function itself,

$$\frac{d}{dt}q^{f(t)} = f'(t)q^{f(t)}\log(q) \quad (14)$$

The method of proof is the same as that of the previous case where $f(t) = t$.

Generally, a quaternion function may be written as

$$q(t) = \cos(\theta(t)) + \hat{u}(t)\sin(\theta(t)) \quad (15)$$

where the angle θ and \hat{u} both vary with t . The derivative is

$$q'(t) = -\sin(\theta(t))\theta'(t) + \hat{u}(t)\cos(\theta(t))\theta'(t) + \hat{u}'(t)\sin(\theta(t)) = q(t)\hat{u}(t)\theta'(t) + \hat{u}'(t)\sin(\theta(t)) \quad (16)$$

Because $-1 = \hat{u}(t)\hat{u}(t)$, we also know that

$$0 = \hat{u}(t)\hat{u}'(t) + \hat{u}'(t)\hat{u}(t) \quad (17)$$

If you write $\hat{u} = xi + yj + zk$ and expand the right-hand side of Equation (17), the equation becomes $xx' + yy' + zz' = 0$. This implies the vectors $\mathbf{u} = (x, y, z)$ and $\mathbf{u}' = (x', y', z')$ are perpendicular. From this discussion, it is easily shown that

$$\hat{u}(t)q'(t) + q'(t)\hat{u}(t) = -2\theta'(t)q(t) \quad (18)$$

Now define

$$h(t) = q(t)^{f(t)} = \cos(f(t)\theta(t)) + \hat{u}(t)\sin(f(t)\theta(t)) \quad (19)$$

where $q(t) = \cos(\theta(t)) + \hat{u}(t)\sin(\theta(t))$. The motivation for the definition is that we know how to compute $q(t)^{f(s)}$ for independent variables s and t , and we want this to be jointly continuous in the sense that $q(t)^{f(t)} = \lim_{s \rightarrow t} q(t)^{f(s)}$. The derivative is

$$h'(t) = [-\sin(f\theta) + \hat{u}\cos(f\theta)](f\theta)' + \hat{u}'\sin(f\theta) = (\hat{u}h)(f\theta)' + \hat{u}'\sin(f\theta) \quad (20)$$

Similar to Equation (18), it may be shown that

$$\hat{u}(t)h'(t) + h'(t)\hat{u}(t) = -2\frac{d[f(t)\theta(t)]}{dt}h(t)$$

Note that this last equation by itself is not enough information to completely determine $h'(t)$, so consider it a sufficient condition for the derivative $h'(t)$.

4 Spherical Linear Interpolation

The previous version of the document had a construction for spherical linear interpolation of two quaternions q_0 and q_1 treated as unit length vectors in 4-dimensional space, the angle θ between them acute. The idea was that $q(t) = c_0(t)q_0 + c_1(t)q_1$ where $c_0(t)$ and $c_1(t)$ are real-valued functions for $0 \leq t \leq 1$ with $c_0(0) = 1$, $c_1(0) = 0$, $c_0(1) = 0$, and $c_1(1) = 1$. The quantity $q(t)$ is required to be a unit vector, so $1 = q(t) \bullet q(t) = c_0(t)^2 + 2\cos(\theta)c_0(t)c_1(t) + c_1(t)^2$. This is the equation of an ellipse that I factored using methods of analytic geometry to obtain formulas for $c_0(t)$ and $c_1(t)$.

A simpler construction uses only trigonometry and solving two equations in two unknowns. As t uniformly varies between 0 and 1, the values $q(t)$ are required to uniformly vary along the circular arc from q_0 to q_1 . That is, the angle between $q(t)$ and q_0 is $\cos(t\theta)$ and the angle between $q(t)$ and q_1 is $\cos((1-t)\theta)$. Dotting the equation for $q(t)$ with q_0 yields

$$\cos(t\theta) = c_0(t) + \cos(\theta)c_1(t)$$

and dotting the equation with q_1 yields

$$\cos((1-t)\theta) = \cos(\theta)c_0(t) + c_1(t).$$

These are two equations in the two unknowns c_0 and c_1 . The solution for c_0 is

$$c_0(t) = \frac{\cos(t\theta) - \cos(\theta)\cos((1-t)\theta)}{1 - \cos^2(\theta)} = \frac{\sin((1-t)\theta)}{\sin(\theta)}.$$

The last equality is obtained by applying double-angle formulas for sine and cosine. By symmetry, $c_1(t) = c_0(1-t)$. Or solve the equations for

$$c_1(t) = \frac{\cos((1-t)\theta) - \cos(\theta)\cos(t\theta)}{1 - \cos^2(\theta)} = \frac{\sin(t\theta)}{\sin(\theta)}.$$

The spherical linear interpolation, abbreviated as *slerp*, is defined by

$$\text{Slerp}(t; q_0, q_1) = \frac{q_0 \sin((1-t)\theta) + q_1 \sin(t\theta)}{\sin \theta} \quad (21)$$

for $0 \leq t \leq 1$.

Although q_1 and $-q_1$ represent the same rotation, the values of $\text{Slerp}(t; q_0, q_1)$ and $\text{Slerp}(t; q_0, -q_1)$ are not the same. It is customary to choose the sign σ on q_1 so that $q_0 \bullet (\sigma q_1) \geq 0$ (the angle between q_0 and σq_1 is acute). This choice avoids extra spinning caused by the interpolated rotations.

For unit quaternions, *slerp* can be written as

$$\text{Slerp}(t; q_0, q_1) = q_0 (q_0^{-1} q_1)^t. \quad (22)$$

The idea is that $q_1 = q_0(q_0^{-1}q_1)$. The term $q_0^{-1}q_1 = \cos \theta + \hat{u} \sin \theta$ where θ is the angle between q_0 and q_1 . The time parameter can be introduced into the angle so that the adjustment of q_0 varies uniformly with over the great arc between q_0 and q_1 . That is, $q(t) = q_0[\cos(t\theta) + \hat{u} \sin(t\theta)] = q_0[\cos \theta + \hat{u} \sin \theta]^t = q_0(q_0^{-1}q_1)^t$.

The derivative of *slerp* in the form of equation (22) is a simple application of equation (13),

$$\text{Slerp}'(t; q_0, q_1) = q_0(q_0^{-1}q_1)^t \log(q_0^{-1}q_1). \quad (23)$$

5 Spherical Cubic Interpolation

Cubic interpolation of quaternions can be achieved using a method described in [2] which has the flavor of bilinear interpolation on a quadrilateral. The evaluation uses an iteration of three *slerps* and is similar to the de Casteljau algorithm (see [3]). Imagine four quaternions p , a , b , and q as the ordered vertices of a quadrilateral. Interpolate c along the “edge” from p to q using *slerp*. Interpolate d along the “edge” from

a to b . Now interpolate the edge interpolations c and d to get the final result e . The end result is denoted *squad* and is given by

$$\text{Squad}(t; p, a, b, q) = \text{Slerp}(2t(1-t); \text{Slerp}(t; p, q), \text{Slerp}(t; a, b)) \quad (24)$$

For unit quaternions we can use equation (22) to obtain a similar formula for *squad*,

$$\text{Squad}(t; p, a, b, q) = \text{Slerp}(t; p, q)(\text{Slerp}(t; p, q)^{-1} \text{Slerp}(t; a, b))^{2t(1-t)} \quad (25)$$

The derivative of *squad* in equation (25) is computed as follows. To simplify the notation, define $U(t) = \text{Slerp}(t; p, q)$ and $V(t) = \text{Slerp}(t; a, b)$. Equation (13) implies $U'(t) = U(t) \log(p^{-1}q)$ and $V'(t) = V(t) \log(a^{-1}b)$. Define $W(t)$, $\hat{\alpha}(t)$, and $\phi(t)$ by

$$W(t) = U(t)^{-1}V(t) = \cos(\phi(t)) + \hat{\alpha}(t) \sin(\phi(t)) \quad (26)$$

It is simple to see that $U(t)W(t) = V(t)$. The derivative of $W(t)$ is implicit in $U(t)W'(t) + U'(t)W(t) = V'(t)$. The *squad* function is then

$$\text{Squad}(t; p, a, b, q) = U(t)W(t)^{2t(1-t)} \quad (27)$$

and its derivative is computed as shown next, using Equation (20),

$$\begin{aligned} \text{Squad}'(t; p, q, a, b) &= \frac{d}{dt} [UW^{2t(1-t)}] \\ &= U(t) \frac{d}{dt} [W(t)^{2t(1-t)}] + U'(t) [W(t)^{2t(1-t)}] \\ &= U(t) \left\{ \hat{\alpha}(t)W(t)^{2t(1-t)} [2t(1-t)\phi'(t) + (2-4t)\phi(t)] + \hat{\alpha}'(t) \sin(2t(1-t)\phi(t)) \right\} \\ &\quad + U'(t)W(t)^{2t(1-t)} \end{aligned} \quad (28)$$

For spline interpolation using *squad* we will need to evaluate the derivative of *squad* at $t = 0$ and $t = 1$. Observe that $U(0) = p$, $U'(0) = p \log(p^{-1}q)$, $U(1) = q$, $U'(1) = q \log(p^{-1}q)$, $V(0) = a$, $V'(0) = a \log(a^{-1}b)$, $V(1) = b$, and $V'(1) = b \log(a^{-1}b)$. Also observe that $\log(W(t)) = \hat{\alpha}(t)\phi(t)$ so that $\log(p^{-1}a) = \log(W(0)) = \hat{\alpha}(0)\phi(0)$ and $\log(q^{-1}b) = \log(W(1)) = \hat{\alpha}(1)\phi(1)$. The derivatives of *squad* at the endpoints are

$$\begin{aligned} \text{Squad}'(0; p, a, b, q) &= U(0) \{ \hat{\alpha}(0)[+2\phi(0)] \} + U'(0) = p[\log(p^{-1}q) + 2\log(p^{-1}a)] \\ \text{Squad}'(1; p, a, b, q) &= U(1) \{ \hat{\alpha}(1)[-2\phi(1)] \} + U'(1) = q[\log(p^{-1}q) - 2\log(q^{-1}b)] \end{aligned} \quad (29)$$

6 Spline Interpolation of Quaternions

Given a sequence of N unit quaternions $\{q_n\}_{n=0}^{N-1}$, we want to build a spline which interpolates those quaternions subject to the conditions that the spline pass through the control points and that the derivatives are continuous. The idea is to choose intermediate quaternions a_n and b_n to allow control of the derivatives at the endpoints of the spline segments. More precisely, let $S_n(t) = \text{Squad}(t; q_n, a_n, b_{n+1}, q_{n+1})$ be the spline segments. By definition of *squad* it is easily shown that

$$S_{n-1}(1) = q_n = S_n(0).$$

To obtain continuous derivatives at the endpoints we need to match the derivatives of two consecutive spline segments,

$$S'_{n-1}(1) = S'_n(0).$$

It can be shown from equation (29) that

$$S'_{n-1}(1) = q_n [\log(q_{n-1}^{-1} q_n) - 2 \log(q_n^{-1} b_n)]$$

and

$$S'_n(0) = q_n [\log(q_n^{-1} q_{n+1}) + 2 \log(q_n^{-1} a_n)].$$

The derivative continuity equation provides one equation in the two unknowns a_n and b_n , so we have one degree of freedom. As suggested in [1], a good choice for the derivative at the control point uses an average T_n of “tangents”, so $S'_{n-1}(1) = q_n T_n = S'_n(0)$ where

$$T_n = \frac{\log(q_n^{-1} q_{n+1}) + \log(q_{n-1}^{-1} q_n)}{2}. \quad (30)$$

We now have two equations to determine a_n and b_n . Some algebra will show that

$$a_n = b_n = q_n \exp\left(-\frac{\log(q_n^{-1} q_{n+1}) + \log(q_n^{-1} q_{n-1})}{4}\right). \quad (31)$$

Thus, $S_n(t) = \text{Squad}(t; q_n, a_n, a_{n+1}, q_{n+1})$.

EXAMPLE. To illustrate the cubic nature of the interpolation, consider a sequence of quaternions whose general term is $q_n = \exp(i\theta_n)$. This is a sequence of complex numbers whose products do commute and for which the usual properties of exponents and logarithms do apply. The intermediate terms are

$$a_n = \exp(-i(\theta_{n+1} - 6\theta_n + \theta_{n-1})/4).$$

Also,

$$\text{Slerp}(t, q_n, q_{n+1}) = \exp(i((1-t)\theta_n + t\theta_{n+1}))$$

and

$$\text{Slerp}(t, a_n, a_{n+1}) = \exp(-i((1-t)(\theta_{n+1} - 6\theta_n + \theta_{n-1}) + t(\theta_{n+2} - 6\theta_{n+1} + \theta_n))/4).$$

Finally,

$$\begin{aligned} \text{Squad}(t, q_n, a_n, a_{n+1}, q_{n+1}) &= \exp([1 - 2t(1-t)][(1-t)\theta_n + t\theta_{n+1}] \\ &\quad - [2t(1-t)/4][(\theta_{n+1} - 6\theta_n + \theta_{n-1}) + t(\theta_{n+2} - 6\theta_{n+1} + \theta_n)]). \end{aligned}$$

The angular cubic interpolation is

$$\phi(t) = -\frac{1}{2}t^2(1-t)\theta_{n+2} + \frac{1}{2}t(2 + 2(1-t) - 3(1-t)^2)\theta_{n+1} + \frac{1}{2}(1-t)(2 + 2t - 3t^2)\theta_n - \frac{1}{2}t(1-t)^2\theta_{n-1}.$$

It can be shown that $\phi(0) = \theta_n$, $\phi(1) = \theta_{n+1}$, $\phi'(0) = (\theta_{n+1} - \theta_{n-1})/2$, and $\phi'(1) = (\theta_{n+2} - \theta_n)/2$. The derivatives at the end points are centralized differences, the average of left and right derivatives as expected.

References

- [1] Ken Shoemake, *Animating rotation with quaternion calculus*, ACM SIGGRAPH 1987, Course Notes 10, Computer Animation: 3-D Motion, Specification, and Control.
- [2] W. Boehm, *On cubics: a survey*, Computer Graphics and Image Processing, vol. 19, pp. 201-226, 1982
- [3] Gerald Farin, *Curves and Surfaces for Computer Aided Geometric Design*, Academic Press, Inc., San Diego CA, 1990