# Perspective Projection of an Ellipse onto a Plane in 3D

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## Contents

# 1 Introduction

This document shows how to compute the perspective projection of a 3D ellipse with respect to an eye point and a plane. If the projection onto the plane exists, the projection can be an ellipse, a parabola, a hyperbola, a line, a ray or a segment. The first three cases occur when the eye point is not in the ellipse plane. The last three cases occur when the eye point is in the ellipse plane.

The *projection plane* is $\boldsymbol{N} \cdot (\boldsymbol{X} - \boldsymbol{P}) = 0$ with origin $\boldsymbol{P}$ and unit-length normal $\boldsymbol{N}$.

The *eye point* is $\boldsymbol{E}$ which is assumed to be on the side of the projection plane to which $\boldsymbol{N}$ is directed. This condition is quantified by $\boldsymbol{N} \cdot (\boldsymbol{E} - \boldsymbol{P}) > 0$. The *eye plane* is $\boldsymbol{N} \cdot (\boldsymbol{X} - \boldsymbol{E}) = 0$ and is parallel to the projection plane.

The 3D ellipse lives in a plane $\boldsymbol{W} \cdot (\boldsymbol{X} - \boldsymbol{C}) = 0$ with origin $\boldsymbol{C}$ and unit-length normal $\boldsymbol{W}$. Moreover, the ellipse has center $\boldsymbol{C}$, unit-length major axis $\boldsymbol{U}_0$ with extent $\ell_0$ and unit-length minor axis $\boldsymbol{U}_1$ with extent $\ell_1$ such that $\ell_0 \geq \ell_1 > 0$. Points on the ellipse are defined by

$$\boldsymbol{X} = \boldsymbol{C} + y_0 \boldsymbol{U}_0 + y_1 \boldsymbol{U}_1 = \boldsymbol{C} + J\boldsymbol{Y} \tag{1}$$

where $(y_0/\ell_0)^2 + (y_1/\ell_1)^2 = 1$, an equation defining a standard ellipse in 2D with center at the origin $(0,0)$, major axis direction $(1,0)$, minor axis direction $(0,1)$ and $\ell_0 \geq \ell_1 > 0$. The matrix $J = [\boldsymbol{U}_0\,\boldsymbol{U}_1]$ is a $3 \times 2$ matrix whose columns are the ellipse axis directions and $\boldsymbol{Y} = [y_0\, y_1]^\mathsf{T}$ is a $2 \times 1$ vector whose rows are the $y$-coordinates. The extreme points along the major axis are $\boldsymbol{C} \pm \ell_0 \boldsymbol{U}_0$ and the extreme points along the minor axis are $\boldsymbol{C} \pm \ell_1 \boldsymbol{U}_1$.

Equation (1) can be converted to a quadratic equation $(\boldsymbol{X} - \boldsymbol{C})^\mathsf{T} A (\boldsymbol{X} - \boldsymbol{C}) = 1$ subject to the constraint $\boldsymbol{W}^\mathsf{T}(\boldsymbol{X} - \boldsymbol{C}) = 0$. Specifically, $y_0 = \boldsymbol{U}_0^\mathsf{T}(\boldsymbol{X} - \boldsymbol{C})$, $y_1 = \boldsymbol{U}_1^\mathsf{T}(\boldsymbol{X} - \boldsymbol{C})$, and the ellipse equation $(y_0/\ell_0)^2 + (y_1/\ell_1)^2 = 1$ becomes

$$\begin{aligned}
1 &= (y_0/\ell_0)^2 + (y_1/\ell_1)^2 \\
&= (\boldsymbol{U}_0^\mathsf{T}(\boldsymbol{X} - \boldsymbol{C})/\ell_0)^2 + (\boldsymbol{U}_1^\mathsf{T}(\boldsymbol{X} - \boldsymbol{C})/\ell_1)^2 \\
&= (\boldsymbol{X} - \boldsymbol{C})^\mathsf{T} \boldsymbol{U}_0 \boldsymbol{U}_0^\mathsf{T} (\boldsymbol{X} - \boldsymbol{C})/\ell_0^2 + (\boldsymbol{X} - \boldsymbol{C})^\mathsf{T} \boldsymbol{U}_1 \boldsymbol{U}_1^\mathsf{T} (\boldsymbol{X} - \boldsymbol{C})/\ell_1^2 \\
&= (\boldsymbol{X} - \boldsymbol{C})^\mathsf{T} \left( \boldsymbol{U}_0 \boldsymbol{U}_0^\mathsf{T}/\ell_0^2 + \boldsymbol{U}_1 \boldsymbol{U}_1^\mathsf{T}/\ell_1^2 \right) (\boldsymbol{X} - \boldsymbol{C}) \\
&= (\boldsymbol{X} - \boldsymbol{C})^\mathsf{T} J L^{-2} J^\mathsf{T} (\boldsymbol{X} - \boldsymbol{C}) \\
&= (\boldsymbol{X} - \boldsymbol{C})^\mathsf{T} A (\boldsymbol{X} - \boldsymbol{C})
\end{aligned} \tag{2}$$

where $L = \mathrm{Diag}(\ell_0, \ell_1)$ is a $2 \times 2$ diagonal matrix, $L^{-1} = \mathrm{Diag}(1/\ell_0, 1/\ell_1)$, $L^{-2} = L^{-1}L^{-1}$ and $A = JL^{-2}J^\mathsf{T}$ is a $3 \times 3$ symmetric matrix of rank 2. The constraint $\boldsymbol{W}^\mathsf{T}(\boldsymbol{X} - \boldsymbol{C}) = 0$ still applies.

# 2 Determining the Projectable Ellipse Points

Projectable points relative to the eye point and projection plane are those points $\boldsymbol{X}$ for which the ray $\boldsymbol{R}(t) = \boldsymbol{E} + t(\boldsymbol{X} - \boldsymbol{E})$ intersects the projection plane. The intersection occurs when $\boldsymbol{N} \cdot (\boldsymbol{R}(t) - \boldsymbol{P}) = 0$ for some $t > 0$. The solution is $t = \boldsymbol{N} \cdot (\boldsymbol{P} - \boldsymbol{E})/\boldsymbol{N} \cdot (\boldsymbol{X} - \boldsymbol{E})$. The numerator is negative because the eye point has the constraint $\boldsymbol{N} \cdot (\boldsymbol{E} - \boldsymbol{P}) > 0$. For $t$ to be positive, the denominator must also be negative; that is, $\boldsymbol{N} \cdot (\boldsymbol{X} - \boldsymbol{E}) < 0$. Therefore, $\boldsymbol{X}$ must lie on the negative side of the eye plane.

The ellipse can be projected onto the normal line $\boldsymbol{P}+s\boldsymbol{N}$, the result either an interval or a single point. The ellipse $(y_0/\ell_0)^2+(y_1/\ell_1)^2=1$ is parameterized by $(y_0,y_1)=(\ell_0\cos\theta,\ell_1\sin\theta)$ for $\theta\in[0,2\pi)$. Substituting the ellipse points $\boldsymbol{X}$ defined in equation (1) into the equation for the normal line,

$$\begin{aligned}\boldsymbol{N}\cdot(\boldsymbol{X}-\boldsymbol{P}) &= \boldsymbol{N}\cdot(\boldsymbol{C}-\boldsymbol{P})+y_0\boldsymbol{N}\cdot\boldsymbol{U}+y_1\boldsymbol{N}\cdot\boldsymbol{V}\\ &= \boldsymbol{N}\cdot(\boldsymbol{C}-\boldsymbol{P})+(\ell_0\boldsymbol{N}\cdot\boldsymbol{U})\cos\theta+(\ell_1\boldsymbol{N}\cdot\boldsymbol{V})\sin\theta\\ &= \boldsymbol{N}\cdot(\boldsymbol{C}-\boldsymbol{P})+r(\theta)\end{aligned}\tag{3}$$

where the last equality defines the function $r(\theta)$ for $\theta\in[0,2\pi)$. The extreme values of $r(\theta)$ occur when $(\cos\theta,\sin\theta)$ is parallel to $(\ell_0\boldsymbol{N}\cdot\boldsymbol{U},\ell_1\boldsymbol{N}\cdot\boldsymbol{V})$,

$$(\cos\theta,\sin\theta)=\pm\frac{(\ell_0\boldsymbol{N}\cdot\boldsymbol{U},\ell_1\boldsymbol{N}\cdot\boldsymbol{V})}{\sqrt{(\ell_0\boldsymbol{N}\cdot\boldsymbol{U})^2+(\ell_1\boldsymbol{N}\cdot\boldsymbol{V})^2}}\tag{4}$$

from which it follows

$$r_{\max}=\sqrt{(\ell_0\boldsymbol{N}\cdot\boldsymbol{U})^2+(\ell_1\boldsymbol{N}\cdot\boldsymbol{V})^2},\ \ r(\theta)\in[-r_{\max},r_{\max}]\tag{5}$$

The projection of the ellipse onto the normal line is the interval $[\boldsymbol{N}\cdot\boldsymbol{\Delta}_{\text{cp}}-r_{\max},\boldsymbol{N}\cdot\boldsymbol{\Delta}_{\text{cp}}+r_{\max}]$, where $\boldsymbol{\Delta}_{\text{cp}}=\boldsymbol{C}-\boldsymbol{P}$. Define also $\boldsymbol{\Delta}_{\text{ep}}=\boldsymbol{E}-\boldsymbol{P}$. Projectable points are classified according to the following list:

1. All ellipse points are projectable when $\boldsymbol{N}\cdot\boldsymbol{\Delta}_{\text{cp}}+r_{\max}<\boldsymbol{N}\cdot\boldsymbol{\Delta}_{\text{ep}}$.
2. None of the ellipse points are projectable when $\boldsymbol{N}\cdot\boldsymbol{\Delta}_{\text{cp}}-r_{\max}\geq\boldsymbol{N}\cdot\boldsymbol{\Delta}_{\text{ep}}$.
3. All ellipse points except for one point are projectable when $\boldsymbol{N}\cdot\boldsymbol{\Delta}_{\text{cp}}+r_{\max}\leq\boldsymbol{N}\cdot\boldsymbol{\Delta}_{\text{ep}}$. The exceptional point corresponds to $\hat{\theta}$ for which $r(\hat{\theta})=r_{\max}$.
4. If the eye line and ellipse intersect in exactly 2 points, those points are not projectable. The ellipse minus the 2 points is a union of disjoint and open elliptical arcs. One arc has points that project onto the normal line in an interval $[\boldsymbol{N}\cdot\boldsymbol{\Delta}_{\text{cp}}-r_{\max},\boldsymbol{N}\cdot\boldsymbol{\Delta}_{\text{ep}})$; these points are all projectable. The other arc has projection interval $(\boldsymbol{N}\cdot\boldsymbol{\Delta}_{\text{ep}},\boldsymbol{N}\cdot\boldsymbol{\Delta}_{\text{cp}}+r_{\max}]$; none of these points are projectable.

The classification of the projection sets is described in the next sections.

## 3 The Eye Point is in the Ellipse Plane

The eye point is in the plane of the ellipse when $\boldsymbol{W}\cdot(\boldsymbol{E}-\boldsymbol{C})=0$. In this case the projection algorithm reduces to the 2-dimensional problem of projecting a 2D ellipse onto a 2D line. The projection is described in Perspective Projection of an Ellipse onto a Line in 2D. The goal in this section is to represent the various objects in a 2-dimensional coordinate system for the plane.

If $\boldsymbol{N}$ and $\boldsymbol{W}$ are parallel, the ellipse is on the eye plane; no points are projectable. Otherwise, $\boldsymbol{N}$ and $\boldsymbol{W}$ are not parallel and the projection plane and ellipse plane intersect in a line $\boldsymbol{L}(s)=s\boldsymbol{D}+\boldsymbol{Q}$, where $\boldsymbol{D}=\boldsymbol{N}\times\boldsymbol{W}/|\boldsymbol{N}\times\boldsymbol{W}|$ is the unit-length line direction and $\boldsymbol{Q}=\boldsymbol{C}+q_0\boldsymbol{U}_0+q_1\boldsymbol{U}_1$ is a point on the line. The line is on the projection plane, so

$$0=\boldsymbol{N}\cdot(\boldsymbol{L}(s)-\boldsymbol{P})=\boldsymbol{N}\cdot(\boldsymbol{Q}-\boldsymbol{P})=\boldsymbol{N}\cdot(\boldsymbol{Q}-\boldsymbol{C})-\boldsymbol{N}\cdot(\boldsymbol{P}-\boldsymbol{C})\tag{6}$$

The equation has one degree of freedom because if $\boldsymbol{Q}$ is on the line of intersection of the ellipse plane and the projection plane, then so is $\boldsymbol{Q} + s\boldsymbol{D}$ for any scalar $s$. Eliminate the degree of freedom by requiring $\boldsymbol{Q} - \boldsymbol{C}$ to have no component in the $\boldsymbol{D}$ direction,

$$0 = \boldsymbol{D} \cdot (\boldsymbol{Q} - \boldsymbol{C}) = q_0 \boldsymbol{D} \cdot \boldsymbol{U}_0 + q_1 \boldsymbol{D} \cdot \boldsymbol{U}_1 \tag{7}$$

Equations (6) and (7) form a linear system of 2 equations in 2 unknowns. The solution is

$$\begin{bmatrix} q_0 \\ q_1 \end{bmatrix} = \frac{\boldsymbol{N} \cdot (\boldsymbol{P} - \boldsymbol{C})}{|\boldsymbol{N} \times \boldsymbol{W}|} \begin{bmatrix} -\boldsymbol{D} \cdot \boldsymbol{U}_1 \\ +\boldsymbol{D} \cdot \boldsymbol{U}_0 \end{bmatrix} \tag{8}$$

The eye point can be written as $\boldsymbol{E} = \boldsymbol{C} + e_0 \boldsymbol{U}_0 + e_1 \boldsymbol{U}_1$, where $e_i = \boldsymbol{U}_i \cdot (\boldsymbol{E} - \boldsymbol{C})$. The line direction is $\boldsymbol{D} = d_0 \boldsymbol{U}_0 + d_1 \boldsymbol{U}_1$, where $d_i = \boldsymbol{U}_i \cdot \boldsymbol{D}$. The ellipse center $\boldsymbol{C}$ is selected to be the 2D origin $(0,0)$. The 2D algorithm for projection of an ellipse onto a line can be applied by choosing the ellipse center $(0,0)$, the major axis direction $(1,0)$ with extent $\ell_0$, the minor axis direction $(0,1)$ with extent $\ell_1$, the eye point $(e_0, e_1)$, the projection line origin $(q_0, q_1)$ and the projection line direction $(d_0, d_1)$. The output of the 2D algorithm contains 2-tuples of the form $(z_0, z_1)$ which must be lifted back to 3D by $\boldsymbol{C} + z_0 \boldsymbol{U}_0 + z_1 \boldsymbol{U}_1$.

# 4 The Eye Point is not in the Ellipse Plane

Projection sets are classified according to the following list.

1. All ellipse points are projectable when $\boldsymbol{N} \cdot \boldsymbol{\Delta}_{\text{cp}} + r_{\max} < \boldsymbol{N} \cdot \boldsymbol{\Delta}_{\text{ep}}$. The projection set is an ellipse.
2. None of the ellipse points are projectable when $\boldsymbol{N} \cdot \boldsymbol{\Delta}_{\text{cp}} - r_{\max} \geq \boldsymbol{N} \cdot \boldsymbol{\Delta}_{\text{ep}}$. The projection set is empty.
3. All ellipse points except for one point are projectable when $\boldsymbol{N} \cdot \boldsymbol{\Delta}_{\text{cp}} + r_{\max} \leq \boldsymbol{N} \cdot \boldsymbol{\Delta}_{\text{ep}}$. The exceptional point corresponds to $\hat{\theta}$ for which $\rho(\hat{\theta}) = r_{\max}$. The projection set is a parabola.
4. If the eye line and ellipse intersect in exactly 2 points, those points are not projectable. The ellipse minus the 2 points is a union of disjoint and open elliptical arcs. One arc has points that project onto the normal line in an interval $[\boldsymbol{N} \cdot \boldsymbol{\Delta}_{\text{cp}} - r_{\max}, \boldsymbol{N} \cdot \boldsymbol{\Delta}_{\text{ep}})$; these points are all projectable. The other arc has projection interval $(\boldsymbol{N} \cdot \boldsymbol{\Delta}_{\text{ep}}, \boldsymbol{N} \cdot \boldsymbol{\Delta}_{\text{cp}} + r_{\max}]$; none of these points are projectable. The projection set is a branch of a hyperbola.

## 4.1 A Parametric Representation of the Projection Set

The projection rays are $\boldsymbol{R}(t) = \boldsymbol{E} + t\boldsymbol{D}$ for unit-length vectors $\boldsymbol{D}$ with constraint $\boldsymbol{N} \cdot \boldsymbol{D} < 0$ and for $t \geq 0$. The dot-product constraint ensures that the ray intersects the projection plane.

We want to determine which rays intersect the ellipse. Define $\boldsymbol{\Delta} = \boldsymbol{E} - \boldsymbol{C}$. The rays must intersect the ellipse plane. Substituting the ray equation into the ellipse plane equation,

$$0 = \boldsymbol{W}^{\mathsf{T}}(\boldsymbol{R}(t) - \boldsymbol{C}) = \boldsymbol{W}^{\mathsf{T}}(t\boldsymbol{D} + \boldsymbol{\Delta}) = t\boldsymbol{W}^{\mathsf{T}}\boldsymbol{D} + \boldsymbol{W}^{\mathsf{T}}\boldsymbol{\Delta} \tag{9}$$

in which case

$$t = -\boldsymbol{W}^{\mathsf{T}}\boldsymbol{\Delta} / \boldsymbol{W}^{\mathsf{T}}\boldsymbol{D} \tag{10}$$

The rays must also intersect the ellipse. Substituting the ray equation into the ellipse equation (2),

$$\begin{aligned} 0 &= (\boldsymbol{R}(t) - \boldsymbol{C})^{\mathsf{T}} A (\boldsymbol{R}(t) - \boldsymbol{C}) - 1 \\ &= (t\boldsymbol{D} + \boldsymbol{\Delta})^{\mathsf{T}} A (t\boldsymbol{D} + \boldsymbol{\Delta}) - 1 \\ &= t^2 \boldsymbol{D}^{\mathsf{T}} A \boldsymbol{D} + 2t\boldsymbol{D}^{\mathsf{T}} A \boldsymbol{\Delta} + \left(\boldsymbol{\Delta}^{\mathsf{T}} A \boldsymbol{\Delta} - 1\right) \end{aligned} \tag{11}$$

Substituting $t$ from equation (10) into equation (11) and multiplying by $(\boldsymbol{W}^{\mathsf{T}}\boldsymbol{D})^2$,

$$\begin{aligned} 0 &= \left(\boldsymbol{W}^{\mathsf{T}}\boldsymbol{\Delta}\right)^2 \boldsymbol{D}^{\mathsf{T}} A \boldsymbol{D} - 2\left(\boldsymbol{W}^{\mathsf{T}}\boldsymbol{D}\right)\left(\boldsymbol{W}^{\mathsf{T}}\boldsymbol{\Delta}\right) \boldsymbol{D}^{\mathsf{T}} A \boldsymbol{\Delta} + \left(\boldsymbol{W}^{\mathsf{T}}\boldsymbol{D}\right)^2 \left(\boldsymbol{\Delta}^{\mathsf{T}} A \boldsymbol{\Delta} - 1\right) \\ &= \boldsymbol{D}^{\mathsf{T}} \left[\left(\boldsymbol{W}^{\mathsf{T}}\boldsymbol{\Delta}\right)^2 A - 2\left(\boldsymbol{W}^{\mathsf{T}}\boldsymbol{\Delta}\right) A \boldsymbol{\Delta} \boldsymbol{W}^{\mathsf{T}} + \left(\boldsymbol{\Delta}^{\mathsf{T}} A \boldsymbol{\Delta} - 1\right) \boldsymbol{W}\boldsymbol{W}^{\mathsf{T}}\right] \boldsymbol{D} \\ &= \boldsymbol{D}^{\mathsf{T}} B \boldsymbol{D} \end{aligned} \tag{12}$$

where the last equality defines the $3 \times 3$ symmetric matrix $B$. This matrix is independent of $\boldsymbol{D}$. As long as there are projectable ellipse points, there must be nonzero solutions $\boldsymbol{D}$ to $\boldsymbol{D}^{\mathsf{T}} B \boldsymbol{D} = 0$. This implies $B$ is not positive definite and not negative definite. Also, $B$ cannot be identically zero.

An eigendecomposition is $B = Q\Gamma Q^{\mathsf{T}}$, where $Q$ is an orthogonal matrix and $\Gamma = \mathrm{Diag}(\gamma_0, \gamma_1, \gamma_2)$ with $\gamma_0 \le \gamma_1 \le \gamma_2$. Define $Q^{\mathsf{T}}\boldsymbol{D} = \boldsymbol{V} = [v_0\ v_1\ v_2]^{\mathsf{T}}$ so that equation (12) becomes

$$0 = \left(Q^{\mathsf{T}}\boldsymbol{D}\right)^{\mathsf{T}} \Gamma \left(Q^{\mathsf{T}}\boldsymbol{D}\right) = \boldsymbol{V}^{\mathsf{T}}\Gamma\boldsymbol{V} = \gamma_0 v_0^2 + \gamma_1 v_1^2 + \gamma_2 v_2^2 \tag{13}$$

where also

$$v_0^2 + v_1^2 + v_2^2 = \boldsymbol{V}^{\mathsf{T}}\boldsymbol{V} = \left(Q^{\mathsf{T}}\boldsymbol{D}\right)^{\mathsf{T}} \left(Q^{\mathsf{T}}\boldsymbol{D}\right) = \boldsymbol{D}^{\mathsf{T}} Q Q^{\mathsf{T}} \boldsymbol{D} = \boldsymbol{D}^{\mathsf{T}}\boldsymbol{D} = 1 \tag{14}$$

The product $QQ^{\mathsf{T}}$ is the identity matrix because $Q$ is an orthogonal matrix. We have 2 quadratic equations in 3 unknowns, so there is 1 degree of freedom. This is to be expected because the ellipse is a curve (1-dimensional object).

The sign configurations for the ordered eigenvalues are shown in Table 1.

**Table 1.** The possible eigenvalue orderings where $\gamma_0 \le \gamma_1 \le \gamma_2$. The first three columns indicate the signs of the eigenvalues. The fourth column is a summary of the eigenvalue constraints corresponding to those signs. The fifth column is the type of projection set.

| Sign($\gamma_0$) | Sign($\gamma_1$) | Sign($\gamma_2$) | constraints | classification |
|---|---|---|---|---|
| − | − | − | $\gamma_0 \le \gamma_1 \le \gamma_2 < 0$ | geometrically not possible |
| − | − | 0 | $\gamma_0 \le \gamma_1 < 0 = \gamma_2$ | geometrically not possible |
| − | − | + | $\gamma_0 \le \gamma_1 < 0 < \gamma_2$ | possible |
| − | 0 | 0 | $\gamma_0 < 0 = \gamma_1 = \gamma_2$ | possible |
| − | 0 | + | $\gamma_0 < 0 = \gamma_1 < \gamma_2$ | possible |
| − | + | + | $\gamma_0 < 0 < \gamma_1 \le \gamma_2$ | possible |
| 0 | 0 | 0 | $\gamma_0 = \gamma_1 = \gamma_2 = 0$ | geometrically not possible |
| 0 | 0 | + | $\gamma_0 = \gamma_1 = 0 < \gamma_2$ | possible |
| 0 | + | + | $\gamma_0 = 0 < \gamma_1 \le \gamma_2$ | $\emptyset$ |
| + | + | + | $0 < \gamma_0 \le \gamma_1 \le \gamma_2$ | geometrically not possible |

The classifications for the eigenvalue orderings are derived in the subsections of this section.

#### 4.1.1 Orders $\gamma_0 \le \gamma_1 \le \gamma_2 < 0$ and $0 < \gamma_0 \le \gamma_1 \le \gamma_2$

Equation (13) is true only when $v_0 = v_1 = v_2 = 0$, which contradicts the unit-length constraint of equation (14). Geometrically, this ordering is not possible.

#### 4.1.2 Order $\gamma_0 \le \gamma_1 < 0 = \gamma_2$

Equation (13) is $\gamma_0 v_0^2 + \gamma_1 v_1^2 = 0$, which is true only when $v_0 = v_1 = 0$. Equation (14) becomes $v_2^2 = 1$. The only 2 solutions are $\boldsymbol{V} = (0, 0, \pm 1)$, which is not geometrically possible. If one ellipse point is projectable, then a continuum of ellipse points is projectable. This in turn implies there must be a continuum of valid vectors $\boldsymbol{V}$ (infinitely many solutions).

#### 4.1.3 Order $\gamma_0 \le \gamma_1 < 0 < \gamma_2$

Substituting $v_2^2 = 1 - v_0^2 - v_1^2$ into equation (13) and manipulating algebraically,

$$\frac{v_0^2}{\gamma_2 - \gamma_1} + \frac{v_1^2}{\gamma_2 - \gamma_0} = \frac{\gamma_2}{(\gamma_2 - \gamma_0)(\gamma_2 - \gamma_1)} \tag{15}$$

The coefficients of $v_0^2$ and $v_1^2$ are positive and the right-hand side is positive, so the equation defines an ellipse that is parameterized by $v_0 = \sqrt{\gamma_2 - \gamma_1}\cos\phi$ and $v_1 = \sqrt{\gamma_2 - \gamma_0}\sin\phi$ for $\phi \in [0, 2\pi)$. The $\boldsymbol{V}$ vectors are

$$\boldsymbol{V}(\phi) = \begin{bmatrix} \sqrt{\gamma_2 - \gamma_1}\cos\phi \\ \sqrt{\gamma_2 - \gamma_0}\sin\phi \\ \pm\sqrt{1 - (\gamma_2 - \gamma_1)\cos^2\phi - (\gamma_2 - \gamma_0)\sin^2\phi} \end{bmatrix} \tag{16}$$

The sign of the $v_2$-component and the angles $\phi$ are chosen so that $0 > \boldsymbol{N} \cdot \boldsymbol{D}(\phi) = \boldsymbol{N}^\mathsf{T} Q \boldsymbol{V}(\phi)$. This equation defines a closed curve of vectors on the unit sphere in 3D, so $\boldsymbol{D}(\phi) = Q\boldsymbol{V}(\phi)$ is a closed curve of vectors on the unit sphere in 3D.

#### 4.1.4 Order $\gamma_0 < 0 = \gamma_1 = \gamma_2$

Equation (13) is $\gamma_0 v_0^2 = 0$, which is true only when $v_0 = 0$. Equation (14) becomes $v_1^2 + v_2^2 = 1$, the equation for a circle that is parameterized by $v_1 = \cos\phi$ and $v_2 = \sin\phi$. The $\boldsymbol{V}$ vectors are

$$\boldsymbol{V}(\phi) = \begin{bmatrix} 0 \\ \cos\phi \\ \sin\phi \end{bmatrix} \tag{17}$$

The angles $\phi$ are chosen so that $0 > \boldsymbol{N} \cdot \boldsymbol{D}(\phi) = \boldsymbol{N}^\mathsf{T} Q \boldsymbol{V}(\phi)$. This equation defines a closed curve of vectors on the unit sphere in 3D, so $\boldsymbol{D}(\phi) = Q\boldsymbol{V}(\phi)$ is a closed curve of vectors on the unit sphere in 3D.

#### 4.1.5 Order $\gamma_0 < 0 = \gamma_1 < \gamma_2$

Substituting $v_2^2 = 1 - v_0^2 - v_1^2$ into equation (13) and manipulating algebraically,

$$\frac{v_0^2}{\gamma_2} + \frac{v_1^2}{\gamma_2 - \gamma_0} = \frac{1}{\gamma_2 - \gamma_0} \tag{18}$$

The coefficients of $v_0^2$ and $v_1^2$ are positive and the right-hand side is positive, so the equation defines an ellipse that is parameterized by $v_0 = \sqrt{\gamma_2}\cos\phi$ and $v_1 = \sqrt{\gamma_2 - \gamma_0}\sin\phi$ for $\phi \in [0, 2\pi)$. The $\boldsymbol{V}$ vectors are

$$\boldsymbol{V}(\phi) = \begin{bmatrix} \sqrt{\gamma_2}\cos\phi \\ \sqrt{\gamma_2 - \gamma_0}\sin\phi \\ \pm\sqrt{1 - \gamma_2\cos^2\phi - (\gamma_2 - \gamma_0)\sin^2\phi} \end{bmatrix} \tag{19}$$

The sign of the $v_2$-component and the angles $\phi$ are chosen so that $0 > \boldsymbol{N} \cdot \boldsymbol{D}(\phi) = \boldsymbol{N}^\mathsf{T} Q \boldsymbol{V}(\phi)$. This equation defines a closed curve of vectors on the unit sphere in 3D, so $\boldsymbol{D}(\phi) = Q\boldsymbol{V}(\phi)$ is a closed curve of vectors on the unit sphere in 3D.

#### 4.1.6 Order $\gamma_0 < 0 < \gamma_1 \le \gamma_2$

Substituting $v_0^2 = 1 - v_1^2 - v_2^2$ into equation (13) and manipulating algebraically,

$$\frac{v_1^2}{\gamma_2 - \gamma_0} + \frac{v_2^2}{\gamma_1 - \gamma_0} = \frac{-\gamma_0}{(\gamma_1 - \gamma_0)(\gamma_2 - \gamma_0)} \tag{20}$$

The coefficients of $v_1^2$ and $v_2^2$ are positive and the right-hand side is positive, so the equation defines an ellipse that is parameterized by $v_1 = \sqrt{\gamma_2 - \gamma_0}\cos\phi$ and $v_2 = \sqrt{\gamma_1 - \gamma_0}\sin\phi$ for $\phi \in [0, 2\pi)$. The $\boldsymbol{V}$ vectors are

$$\boldsymbol{V}(\phi) = \begin{bmatrix} \pm\sqrt{1 - (\gamma_2 - \gamma_0)\cos^2\phi - (\gamma_1 - \gamma_0)\sin^2\phi} \\ \sqrt{\gamma_2 - \gamma_0}\cos\phi \\ \sqrt{\gamma_1 - \gamma_0}\sin\phi \end{bmatrix} \tag{21}$$

The sign of the $v_0$-component and the angles $\phi$ are chosen so that $0 > \boldsymbol{N} \cdot \boldsymbol{D}(\phi) = \boldsymbol{N}^{\mathsf{T}} Q \boldsymbol{V}(\phi)$. This equation defines a closed curve of vectors on the unit sphere in 3D, so $\boldsymbol{D}(\phi) = Q\boldsymbol{V}(\phi)$ is a closed curve of vectors on the unit sphere in 3D.

#### 4.1.7 Order $\gamma_0 = \gamma_1 = 0 < \gamma_2$

Equation (13) is $\gamma_2 v_2^2 = 0$, which is true only when $v_2 = 0$. Equation (14) becomes $v_0^2 + v_1^2 = 1$, the equation for a circle that is parameterized by $v_0 = \cos\phi$ and $v_1 = \sin\phi$. The $\boldsymbol{V}$ vectors are

$$\boldsymbol{V}(\phi) = \begin{bmatrix} \cos\phi \\ \sin\phi \\ 0 \end{bmatrix} \tag{22}$$

The angles $\phi$ are chosen so that $0 > \boldsymbol{N} \cdot \boldsymbol{D}(\phi) = \boldsymbol{N}^{\mathsf{T}} Q \boldsymbol{V}(\phi)$. This equation defines a closed curve of vectors on the unit sphere in 3D, so $\boldsymbol{D}(\phi) = Q\boldsymbol{V}(\phi)$ is a closed curve of vectors on the unit sphere in 3D.

#### 4.1.8 Order $\gamma_0 = 0 < \gamma_1 \le \gamma_2$

Substituting $v_2^2 = 1 - v_0^2 - v_1^2$ into equation (13) and manipulating algebraically,

$$\frac{v_0^2}{\gamma_2 - \gamma_1} + \frac{v_1^2}{\gamma_2} = \frac{1}{\gamma_2 - \gamma_1} \tag{23}$$

The coefficients of $v_0^2$ and $v_1^2$ are positive and the right-hand side is positive, so the equation defines an ellipse that is parameterized by $v_0 = \sqrt{\gamma_2 - \gamma_1}\cos\phi$ and $v_1 = \sqrt{\gamma_2}\sin\phi$ for $\phi \in [0, 2\pi)$. The $\boldsymbol{V}$ vectors are

$$\boldsymbol{V}(\phi) = \begin{bmatrix} \sqrt{\gamma_2 - \gamma_1}\cos\phi \\ \sqrt{\gamma_2}\sin\phi \\ \pm\sqrt{1 - (\gamma_2 - \gamma_1)\cos^2\phi - \gamma_2\sin^2\phi} \end{bmatrix} \tag{24}$$

The sign of the $v_2$-component and the angles $\phi$ are chosen so that $0 > \boldsymbol{N} \cdot \boldsymbol{D}(\phi) = \boldsymbol{N}^{\mathsf{T}} Q \boldsymbol{V}(\phi)$. This equation defines a closed curve of vectors on the unit sphere in 3D, so $\boldsymbol{D}(\phi) = Q\boldsymbol{V}(\phi)$ is a closed curve of vectors on the unit sphere in 3D.

## 4.2 An Algebraic Representation of the Projection Set

The idea is to construct an algebraic equation for the elliptical cone that contains the eye point and the 3D ellipse. The intersection of the elliptical cone with the projection plane is the projection set.

Let the 3D ellipse center be $\boldsymbol{C}_e$, which is a point on the ellipse plane. The major axis direction is the unit-length vector $\boldsymbol{U}_e$ with extent $\ell_0$. The minor axis direction is the unit-length vector $\boldsymbol{V}_e$ with extent $\ell_1$. Let the ellipse plane normal be the unit-length vector $\boldsymbol{N}_e = \boldsymbol{U}_e \times \boldsymbol{V}_e$. The set $\{\boldsymbol{U}_e, \boldsymbol{V}_e, \boldsymbol{N}_e\}$ is a right-handed orthonormal basis. The ellipse consists of 3D points

$$\boldsymbol{X}_e = \boldsymbol{C}_e + y_0\boldsymbol{U}_e + y_1\boldsymbol{V}_e = \boldsymbol{C}_e + J_e\boldsymbol{Y}_e \tag{25}$$

where $(y_0/\ell_0)^2 + (y_1/\ell_1)^2 = 1$. The matrix $J_e = [\boldsymbol{U}_e\ \boldsymbol{V}_e]$ is a $3 \times 2$ matrix whose columns are $\boldsymbol{U}_e$ and $\boldsymbol{V}_e$ and the vector $\boldsymbol{Y}_e = [y_0\ y_1]^{\mathsf{T}}$ is the $2 \times 1$ column vector whose rows are the components of $(y_0, y_1)$.

Given an eye point $\boldsymbol{E}$, we want to perspectively project the ellipse onto another plane. That plane has unit-length normal $\boldsymbol{N}_p$ and origin point $\boldsymbol{C}_p$. We may choose the normal so that $\boldsymbol{E}$ is on the side of the plane to which $\boldsymbol{N}_p$ is directed; that is, $\boldsymbol{N}_p \cdot (\boldsymbol{E} - \boldsymbol{C}_p) > 0$. If $\{\boldsymbol{U}_p, \boldsymbol{V}_p, \boldsymbol{N}_p\}$ is a right-handed orthonormal set, then points on the projection plane are

$$\boldsymbol{X}_p = \boldsymbol{C}_p + y_2\boldsymbol{U}_p + y_3\boldsymbol{V}_p = \boldsymbol{C}_p + J_p\boldsymbol{Y}_p \tag{26}$$

The matrix $J_p = [\boldsymbol{U}_p\ \boldsymbol{V}_p]$ is a $3 \times 2$ matrix whose columns are $\boldsymbol{U}_p$ and $\boldsymbol{V}_p$ and the vector $\boldsymbol{Y}_p = [y_2\ y_3]^{\mathsf{T}}$ is the $2 \times 1$ column vector whose rows are the components of $(y_2, y_3)$. We will determine a quadratic equation in the components of $\boldsymbol{Y}_p$ that define the projected ellipse.

### 4.2.1 Right Elliptical Cones

A standard right elliptical cone is represented algebraically as

$$\left(\frac{y_2}{e_2} - 1\right)^2 = \left(\frac{y_0}{d_0}\right)^2 + \left(\frac{y_1}{d_1}\right)^2 \tag{27}$$

where the cone vertex is located at $(0, 0, e_2)$ with $e_2 > 0$. The cone vertex is a solution to equation (27). When $y_2 = 0$, the elliptical cone equation reduces to the standard ellipse equation $(y_0/\ell_0)^2 + (y_1/\ell_1)^2 = 1$. Therefore, the standard ellipse points are on the elliptical cone, which means that these points are the intersection of the elliptical cone with the $y_0y_1$-plane.

### 4.2.2 Skewed Elliptical Cones

The use of the term right elliptical refers to the vector from ellipse origin to cone vertex is perpendicular to the ellipse plane. We can define an elliptical cone generally by allowing the cone vertex to be at a point $\bar{\boldsymbol{E}} = (\bar{e}_0, \bar{e}_1, \bar{e}_2)$ with $\bar{e}_2 > 0$ and the first two components not necessarily zero. The cone axis passes through $\bar{\boldsymbol{E}}$ and the origin $(0, 0, 0)$ but is no longer perpendicular to the ellipse plane. We wish to have an algebraic representation of this cone. Let us first look at a derivation that leads to equation (27).

Define $Q(\boldsymbol{Y}) = \boldsymbol{Y}^{\mathsf{T}}\bar{A}\boldsymbol{Y} + \bar{\boldsymbol{B}}^{\mathsf{T}}\boldsymbol{Y} + \bar{c}$, where as a 3-tuple, $\boldsymbol{Y} = (y_0, y_1, y_2)$. The $3 \times 3$ matrix $\bar{A}$ is symmetric, $\bar{\boldsymbol{B}}$ is a $3 \times 1$ vector, and $\bar{c}$ is a scalar. The right elliptical cone is represented by the quadratic equation $Q(\boldsymbol{Y}) = 0$ for some choice of $\bar{A}$, $\bar{\boldsymbol{B}}$, and $\bar{c}$. Observe that the choices involve 10 parameters, 6 for $\bar{A} = [\bar{a}_{ij}]$, 3 for $\bar{\boldsymbol{B}} = [\bar{b}_j]$, and 1 for $\bar{c}$.

- When $y_2 = 0$, $Q(\boldsymbol{Y}) = 0$ must reduce to the standard ellipse equation. Replacing $y_2 = 0$ in the quadratic equation,

$$\begin{bmatrix} y_0 & y_1 \end{bmatrix} \begin{bmatrix} \bar{a}_{00} & \bar{a}_{01} \\ \bar{a}_{01} & \bar{a}_{11} \end{bmatrix} \begin{bmatrix} y_0 \\ y_1 \end{bmatrix} + \begin{bmatrix} \bar{b}_0 & \bar{b}_1 \end{bmatrix} \begin{bmatrix} y_0 \\ y_1 \end{bmatrix} + \bar{c} = 0$$

  To obtain the standard ellipse equation, we need $\bar{a}_{00} = 1/d_0^2$, $\bar{a}_{01} = 0$, $\bar{a}_{11} = 1/d_1^2$, $\bar{b}_0 = 0$, $\bar{b}_1 = 0$, and $\bar{c} = -1$.

- The surface normal at the cone vertex is degenerate, so the gradient must be the zero vector: $\mathbf{0} = \nabla Q(\bar{\boldsymbol{E}}) = 2\bar{A}\bar{\boldsymbol{E}} + \bar{\boldsymbol{B}}$. Using the information from item 1 and this constraint, we have $\bar{a}_{02} = -\bar{e}_0/(\bar{e}_2 d_0^2)$, $\bar{a}_{12} = -\bar{e}_1/(\bar{e}_2 d_1^2)$, and $2\bar{e}_2^2\bar{a}_{22} + \bar{e}_2\bar{b}_2 = 2(\bar{e}_0^2/d_0^2 + \bar{e}_1^2/d_1^2)$.

- The cone vertex solves $Q(\bar{\boldsymbol{E}}) = 0$, which implies $\bar{e}_2^2\bar{a}_{22} + \bar{e}_2\bar{b}_2 = 1 + (\bar{e}_0^2/d_0^2 + \bar{e}_1^2/d_1^2)$.

- Items 2 and 3 are solved for $\bar{a}_{22} = (\bar{e}_0^2/d_0^2 + \bar{e}_1^2/d_1^2 - 1)/\bar{e}_2^2$ and $\bar{b}_2 = 2/\bar{e}_2$.

We have determined $\bar{A}$, $\bar{\boldsymbol{B}}$, and $\bar{c}$, namely,

$$\bar{A} = \begin{bmatrix} 1/d_0^2 & 0 & -\bar{e}_0/(\bar{e}_2 d_0^2) \\ 0 & 1/d_1^2 & -\bar{e}_1/(\bar{e}_2 d_1^2) \\ -\bar{e}_0/(\bar{e}_2 d_0^2) & -\bar{e}_1/(\bar{e}_2 d_1^2) & (\bar{e}_0^2/d_0^2 + \bar{e}_1^2/d_1^2 - 1)/\bar{e}_2^2 \end{bmatrix}, \quad \bar{\boldsymbol{B}} = \begin{bmatrix} 0 \\ 0 \\ 2/\bar{e}_2 \end{bmatrix}, \quad \bar{c} = -1 \tag{28}$$

Some factorization leads to the equation for the elliptical cone,

$$\left(\frac{y_2}{\bar{e}_2} - 1\right)^2 = \left(\frac{y_0}{d_0} - \frac{\bar{e}_0}{d_0}\frac{y_2}{\bar{e}_2}\right)^2 + \left(\frac{y_1}{d_1} - \frac{\bar{e}_1}{d_1}\frac{y_2}{\bar{e}_2}\right)^2 \tag{29}$$

#### 4.2.3 General Elliptical Cones

In the most general case, the ellipse is not necessary in a coordinate-axis plane. Equation (25) is the equation for ellipse points, where $(y_0/\ell_0)^2 + (y_1/\ell_1)^2 = 1$. We want to determine $A$, $\boldsymbol{B}$ and $c$ for which $Q(\boldsymbol{X}) = \boldsymbol{X}^\mathsf{T} A\boldsymbol{X} + \boldsymbol{B}^\mathsf{T}\boldsymbol{X} + c = 0$ implicitly defines the elliptical cone with vertex $\boldsymbol{E}$ and contains the specified ellipse points $\boldsymbol{X}_e$.

First, the ellipse points must satisfy

$$\begin{aligned} 0 &= Q(\boldsymbol{X}_e) \\ 0 &= \boldsymbol{X}_e^\mathsf{T} A\boldsymbol{X}_e + \boldsymbol{B}^\mathsf{T}\boldsymbol{X}_e + c \\ &= (\boldsymbol{C}_e + J_e\boldsymbol{Y}_e)^\mathsf{T} A\,(\boldsymbol{C}_e + J_e\boldsymbol{Y}_e) + \boldsymbol{B}^\mathsf{T}(\boldsymbol{C}_e + J_e\boldsymbol{Y}_e) + c \\ &= \boldsymbol{Y}_e^\mathsf{T}\left(J_e^\mathsf{T} A J_e\right)\boldsymbol{Y}_e + \left(J_e^\mathsf{T}(\boldsymbol{B} + 2A\boldsymbol{C}_e)\right)^\mathsf{T}\boldsymbol{Y}_e + \left(\boldsymbol{C}_e^\mathsf{T} A\boldsymbol{C}_e + \boldsymbol{B}^\mathsf{T}\boldsymbol{C}_e + c\right) \end{aligned} \tag{30}$$

Because $\boldsymbol{Y}_e$ is a solution to the standard ellipse equation,

$$J_e^\mathsf{T} A J_e = \mathrm{Diag}(1/d_0^2, 1/d_1^2), \quad J_e^\mathsf{T}(\boldsymbol{B} + 2A\boldsymbol{C}_e) = (0,0), \quad \boldsymbol{C}_e^\mathsf{T} A\boldsymbol{C}_e + \boldsymbol{B}^\mathsf{T}\boldsymbol{C}_e + c = -1 \tag{31}$$

which is a set of 6 linear equations in the unknowns. Second, the gradient of $Q$ must be the zero vector at the cone vertex,

$$\nabla Q(\boldsymbol{E}) = 2A\boldsymbol{E} + \boldsymbol{B} = \boldsymbol{0} \tag{32}$$

which is a set of 3 linear equations in the unknowns. Third, the cone vertex is a solution to the quadratic equation,

$$Q(\boldsymbol{E}) = \boldsymbol{E}^{\mathsf{T}} A\boldsymbol{E} + \boldsymbol{B}^{\mathsf{T}}\boldsymbol{E} + c = 0 \tag{33}$$

which is 1 linear equation in the unknowns. Altogether we have 10 linear equations in the 10 unknowns, allowing us to use a linear system solver to obtain the solution.

#### 4.2.4 Converting General Cones to Skewed Cones

Alternatively, we may translate the ellipse center to the origin and then rotate the ellipse plane into a coordinate plane, transforming the problem to the one we know has the representation in equation (29). Let $R_e = [\boldsymbol{U}_e\, \boldsymbol{V}_e\, \boldsymbol{N}_e]$ be the rotation matrix whose columns are the right-handed orthonormal basis mentioned previously. A point $\boldsymbol{X}$ in the current coordinate system is related to a point $\boldsymbol{Y}$ in the coordinate system of the skewed cone via the transformations

$$\boldsymbol{X} = \boldsymbol{C}_e + R_e\boldsymbol{Y}, \;\; \boldsymbol{Y} = R_e^{\mathsf{T}}(\boldsymbol{X} - \boldsymbol{C}_e) \tag{34}$$

The eye point $\boldsymbol{E}$ is transformed to an eye point $\bar{\boldsymbol{E}} = R_e^{\mathsf{T}}(\boldsymbol{E} - \boldsymbol{C}_e) = (\bar{e}_0, \bar{e}_1, \bar{e}_2)$. Equation (29) provides the ellipse representation

$$\left(\frac{y_2}{\bar{e}_2} - 1\right)^2 = \left(\frac{y_0}{d_0} - \frac{\bar{e}_0}{d_0}\frac{y_2}{\bar{e}_2}\right)^2 + \left(\frac{y_1}{d_1} - \frac{\bar{e}_1}{d_1}\frac{y_2}{\bar{e}_2}\right)^2 \tag{35}$$

In summary, the vertex of the general elliptical cone is $\boldsymbol{E}$. The ellipse consists of points $\boldsymbol{X}_e = \boldsymbol{C}_e + y_0\boldsymbol{U}_e + y_1\boldsymbol{V}_e$, where $(y_0, y_1)$ is a solution to equation (35) with $\bar{\boldsymbol{E}} = R_e^{\mathsf{T}}(\boldsymbol{E} - \boldsymbol{C}_e)$ and $R_e = [\boldsymbol{U}_e\, \boldsymbol{V}_e\, \boldsymbol{N}_e]$.

For later use, we require the quadratic equation in the transformed coordinate system,

$$\begin{aligned} 0 &= \boldsymbol{X}^{\mathsf{T}} A\boldsymbol{X} + \boldsymbol{B}^{\mathsf{T}}\boldsymbol{X} + c \\ &= (\boldsymbol{C}_e + R_e\boldsymbol{Y})^{\mathsf{T}} A\,(\boldsymbol{C}_e + R_e\boldsymbol{Y}) + \boldsymbol{B}^{\mathsf{T}}(\boldsymbol{C}_e + R_e\boldsymbol{Y}) + c \\ &= \boldsymbol{Y}^{\mathsf{T}}\left(R_e^{\mathsf{T}} A R_e\right)\boldsymbol{Y} + \left(R_e^{\mathsf{T}}(\boldsymbol{B} + 2A\boldsymbol{C}_e)\right)^{\mathsf{T}}\boldsymbol{Y} + \left(\boldsymbol{C}_e^{\mathsf{T}} A\boldsymbol{C}_e + \boldsymbol{B}^{\mathsf{T}}\boldsymbol{C}_e + c\right) \\ &= \boldsymbol{Y}^{\mathsf{T}}\bar{A}\boldsymbol{Y} + \bar{\boldsymbol{B}}^{\mathsf{T}}\boldsymbol{Y} + \bar{c} \end{aligned} \tag{36}$$

where the last equality defines the $3 \times 3$ matrix $\bar{A}$, the $3 \times 1$ vector $\bar{\boldsymbol{B}}$, and the scalar $\bar{c}$; that is,

$$\bar{A} = R_e^{\mathsf{T}} A R_e, \;\; \bar{\boldsymbol{B}} = R_e^{\mathsf{T}}(\boldsymbol{B} + 2A\boldsymbol{C}_e), \;\; \bar{c} = \boldsymbol{C}_e^{\mathsf{T}} A\boldsymbol{C}_e + \boldsymbol{B}^{\mathsf{T}}\boldsymbol{C}_e + c \tag{37}$$

where $\bar{A}$, $\bar{\boldsymbol{B}}$ and $\bar{c}$ are constructed using equation (28). This determines the coefficients of the $\boldsymbol{X}$-quadratic; define $\bar{\boldsymbol{C}}_e = R_e^{\mathsf{T}}\boldsymbol{C}_e$; then

$$A = R_e\bar{A}R_e^{\mathsf{T}}, \;\; \boldsymbol{B} = R_e(\bar{\boldsymbol{B}} - 2\bar{A}\bar{\boldsymbol{C}}_e), \;\; c = \bar{\boldsymbol{C}}_e^{\mathsf{T}}\bar{A}\bar{\boldsymbol{C}}_e - \bar{\boldsymbol{B}}^{\mathsf{T}}\bar{\boldsymbol{C}}_e + \bar{c} \tag{38}$$

#### 4.2.5 Projection of the Ellipse to Another Plane

We now have a representation of the general elliptical cone that contains the eye point $\boldsymbol{E}$ and the user-specified ellipse: $\boldsymbol{X}^{\mathsf{T}} A \boldsymbol{X} + \boldsymbol{B}^{\mathsf{T}} \boldsymbol{X} + c = 0$, where $A$, $\boldsymbol{B}$ and $c$ are defined by equation (38). The quantities $\bar{A}$, $\bar{\boldsymbol{B}}$ and $\bar{c}$ are determined using $\bar{\boldsymbol{E}} = R_e^{\mathsf{T}}(\boldsymbol{E} - \boldsymbol{C}_e)$ in the skewed ellipse equation (28).

We have the projection plane, whose origin is the point $\boldsymbol{C}_p$ and unit-length normal is $\boldsymbol{N}_p$, and a right-handed orthonormal basis $\{\boldsymbol{U}_p, \boldsymbol{V}_p, \boldsymbol{N}_p\}$ that may be used to represent points relative to this plane. The plane points are specified by equation (26). To compute the intersection of cone and plane, substitute the plane points into the quadratic equation for the elliptical cone,

$$
\begin{aligned}
0 &= \boldsymbol{X}_p^{\mathsf{T}} A \boldsymbol{X}_p + \boldsymbol{B}^{\mathsf{T}} \boldsymbol{X}_p + c \\
&= (\boldsymbol{C}_p + J_p \boldsymbol{Y}_p)^{\mathsf{T}} A (\boldsymbol{C}_p + J_p \boldsymbol{Y}_p) + \boldsymbol{B}^{\mathsf{T}} (\boldsymbol{C}_p + J_p \boldsymbol{Y}_p) + c \\
&= \boldsymbol{Y}_p^{\mathsf{T}} \left(J_p^{\mathsf{T}} A J_p\right) \boldsymbol{Y}_p + \left(J_p^{\mathsf{T}} (\boldsymbol{B} + 2A\boldsymbol{C}_p)\right)^{\mathsf{T}} \boldsymbol{Y}_p + \left(\boldsymbol{C}_p^{\mathsf{T}} A \boldsymbol{C}_p + \boldsymbol{B}^{\mathsf{T}} \boldsymbol{C}_p + c\right) \\
&= \boldsymbol{Y}_p^{\mathsf{T}} \hat{A} \boldsymbol{Y}_p + \hat{\boldsymbol{B}}^{\mathsf{T}} \boldsymbol{Y}_p + \hat{c}
\end{aligned}
\tag{39}
$$

where the last equality defines $\hat{A}$, $\hat{\boldsymbol{B}}$ and $\hat{c}$; that is,

$$
\hat{A} = J_p^{\mathsf{T}} A J_p, \quad \hat{\boldsymbol{B}} = J_p^{\mathsf{T}} (\boldsymbol{B} + 2A\boldsymbol{C}_p), \quad \hat{c} = \boldsymbol{C}_p^{\mathsf{T}} A \boldsymbol{C}_p + \boldsymbol{B}^{\mathsf{T}} \boldsymbol{C}_p + c \tag{40}
$$

The projection set is an ellipse when $\hat{A}$ all eigenvalues are positive or all eigenvalues are negative. It is a parabola when $\hat{A}$ has a zero eigenvalue. It is a branch of a hyperbola when $\hat{A}$ has a positive eigenvalue and a negative eigenvalue. To determine which branch, project any projectable point on the 3D ellipse to the projection plane and use the corresponding branch.