# Perspective Projection of an Ellipse onto a Line in 2D

David Eberly, Geometric Tools, Redmond WA 98052
https://www.geometrictools.com/



Created: October 29, 2021

## Contents

# 1 Introduction

This document shows how to compute the perspective projection of a 2D ellipse with respect to an eye point and a line. If the projection onto the line exists, the projection can be a line, a ray or a segment.

The *projection line* is $\boldsymbol{N} \cdot (\boldsymbol{X} - \boldsymbol{P}) = 0$ with origin $\boldsymbol{P} = (p_0, p_1)$, unit-length normal $\boldsymbol{N} = (n_0, n_1)$ and $\boldsymbol{X} = (x_0, x_1)$. The line direction is $\boldsymbol{N}^\perp = (n_1, -n_0)$.

The *eye point* is $\boldsymbol{E} = (e_0, e_1)$, which is assumed to be on the side of the projection line to which $\boldsymbol{N}$ is directed. This condition is quantified by $\boldsymbol{N} \cdot (\boldsymbol{E} - \boldsymbol{P}) > 0$. The *eye line* is $\boldsymbol{N} \cdot (\boldsymbol{X} - \boldsymbol{E}) = 0$ and is parallel to the projection line.

The ellipse has center $\boldsymbol{C} = (c_0, c_1)$, unit-length major axis direction $\boldsymbol{U}_0 = (u_{00}, u_{01})$ with extent $\ell_0$, and unit-length minor axis direction $\boldsymbol{U}_1 = (u_{10}, u_{11}) = (-u_{01}, u_{00})$ with extent $\ell_1$ such that $\ell_0 \geq \ell_1 > 0$. Points on the ellipse are defined by

$$\boldsymbol{X} = \boldsymbol{C} + y_0 \boldsymbol{U}_0 + y_1 \boldsymbol{U}_1 = \boldsymbol{C} + J\boldsymbol{Y} \tag{1}$$

where $(y_0/\ell_0)^2 + (y_1/\ell_1)^2 = 1$, an equation defining a standard ellipse in 2D with center at the origin $(0,0)$, major axis direction $(1,0)$, minor axis direction $(0,1)$ and $\ell_0 \geq \ell_1 > 0$. The matrix $J = [\boldsymbol{U}_0 \, \boldsymbol{U}_1]$ is a $2 \times 2$ rotation matrix whose columns are the ellipse axis directions and $\boldsymbol{Y} = [y_0 \, y_1]^\mathsf{T}$ is a $2 \times 1$ vector whose rows are the $y$-coordinates. The extreme points along the major axis are $\boldsymbol{C} \pm \ell_0 \boldsymbol{U}_0$ and the extreme points along the minor axis are $\boldsymbol{C} \pm \ell_1 \boldsymbol{U}_1$.

Equation (1) can be converted to a quadratic equation $(\boldsymbol{X}-\boldsymbol{C})^\mathsf{T} A (\boldsymbol{X}-\boldsymbol{C}) = 1$. Specifically, $y_0 = \boldsymbol{U}_0^\mathsf{T}(\boldsymbol{X}-\boldsymbol{C})$, $y_1 = \boldsymbol{U}_1^\mathsf{T}(\boldsymbol{X} - \boldsymbol{C})$, and the ellipse equation $(y_0/\ell_0)^2 + (y_1/\ell_1)^2 = 1$ becomes

$$\begin{aligned}
1 &= (y_0/\ell_0)^2 + (y_1/\ell_1)^2 \\
&= (\boldsymbol{U}_0^\mathsf{T}(\boldsymbol{X} - \boldsymbol{C})/\ell_0)^2 + (\boldsymbol{U}_1^\mathsf{T}(\boldsymbol{X} - \boldsymbol{C})/\ell_1)^2 \\
&= (\boldsymbol{X} - \boldsymbol{C})^\mathsf{T} \boldsymbol{U}_0 \boldsymbol{U}_0^\mathsf{T} (\boldsymbol{X} - \boldsymbol{C})/\ell_0^2 + (\boldsymbol{X} - \boldsymbol{C})^\mathsf{T} \boldsymbol{U}_1 \boldsymbol{U}_1^\mathsf{T} (\boldsymbol{X} - \boldsymbol{C})/\ell_1^2 \\
&= (\boldsymbol{X} - \boldsymbol{C})^\mathsf{T} \left( \boldsymbol{U}_0 \boldsymbol{U}_0^\mathsf{T}/\ell_0^2 + \boldsymbol{U}_1 \boldsymbol{U}_1^\mathsf{T}/\ell_1^2 \right) (\boldsymbol{X} - \boldsymbol{C}) \\
&= (\boldsymbol{X} - \boldsymbol{C})^\mathsf{T} J L^{-2} J^\mathsf{T} (\boldsymbol{X} - \boldsymbol{C}) \\
&= (\boldsymbol{X} - \boldsymbol{C})^\mathsf{T} A (\boldsymbol{X} - \boldsymbol{C})
\end{aligned} \tag{2}$$

where $L = \mathrm{Diag}(\ell_0, \ell_1)$, $L^{-1} = \mathrm{Diag}(1/\ell_0, 1/\ell_1)$, $L^{-2} = L^{-1} L^{-1}$ and $A = J L^{-2} J^\mathsf{T}$ is a symmetric matrix.

# 2 Determining the Projectable Ellipse Points

Projectable points relative to the eye point and projection line are those points $\boldsymbol{X}$ for which the ray $\boldsymbol{R}(t) = \boldsymbol{E} + t(\boldsymbol{X} - \boldsymbol{E})$ intersects the projection line. The intersection occurs when $\boldsymbol{N} \cdot (\boldsymbol{R}(t) - \boldsymbol{P}) = 0$ for some $t > 0$. The solution is $t = \boldsymbol{N} \cdot (\boldsymbol{P} - \boldsymbol{E}) / \boldsymbol{N} \cdot (\boldsymbol{X} - \boldsymbol{E})$. The numerator is negative because the eye point has the constraint $\boldsymbol{N} \cdot (\boldsymbol{E} - \boldsymbol{P}) > 0$. For $t$ to be positive, the denominator must also be negative; that is, $\boldsymbol{N} \cdot (\boldsymbol{X} - \boldsymbol{E}) < 0$. Therefore, $\boldsymbol{X}$ must lie on the negative side of the eye line.

The ellipse can be projected onto the normal line $\boldsymbol{P} + s\boldsymbol{N}$, the result either an interval $s$-interval. The ellipse $(y_0/\ell_0)^2 + (y_1/\ell_1)^2 = 1$ is parameterized by $(y_0, y_1) = (\ell_0 \cos\theta, \ell_1 \sin\theta)$ for $\theta \in [0, 2\pi)$. Substituting ellipse points $\boldsymbol{X}$ defined in equation (1) into the equation for the normal line,

$$\begin{aligned} \boldsymbol{N} \cdot (\boldsymbol{X} - \boldsymbol{P}) &= \boldsymbol{N} \cdot (\boldsymbol{C} - \boldsymbol{P}) + y_0 \boldsymbol{N} \cdot \boldsymbol{U} + y_1 \boldsymbol{N} \cdot \boldsymbol{V} \\ &= \boldsymbol{N} \cdot (\boldsymbol{C} - \boldsymbol{P}) + (\ell_0 \boldsymbol{N} \cdot \boldsymbol{U}) \cos\theta + (\ell_1 \boldsymbol{N} \cdot \boldsymbol{V}) \sin\theta \\ &= \boldsymbol{N} \cdot (\boldsymbol{C} - \boldsymbol{P}) + r(\theta) \end{aligned} \tag{3}$$

where the last equality defines the function $r(\theta)$ for $\theta \in [0, 2\pi)$. The extreme values of $r(\theta)$ occur when $(\cos\theta, \sin\theta)$ is parallel to $(\ell_0 \boldsymbol{N} \cdot \boldsymbol{U}, \ell_1 \boldsymbol{N} \cdot \boldsymbol{V})$,

$$(\cos\theta, \sin\theta) = \pm \frac{(\ell_0 \boldsymbol{N} \cdot \boldsymbol{U}, \ell_1 \boldsymbol{N} \cdot \boldsymbol{V})}{\sqrt{(\ell_0 \boldsymbol{N} \cdot \boldsymbol{U})^2 + (\ell_1 \boldsymbol{N} \cdot \boldsymbol{V})^2}} \tag{4}$$

from which it follows

$$r_{\max} = \sqrt{(\ell_0 \boldsymbol{N} \cdot \boldsymbol{U})^2 + (\ell_1 \boldsymbol{N} \cdot \boldsymbol{V})^2}, \quad r(\theta) \in [-r_{\max}, r_{\max}] \tag{5}$$

The projection of the ellipse onto the normal line is the interval $[\boldsymbol{N} \cdot \boldsymbol{\Delta}_{\text{cp}} - r_{\max}, \boldsymbol{N} \cdot \boldsymbol{\Delta}_{\text{cp}} + r_{\max}]$, where $\boldsymbol{\Delta}_{\text{cp}} = \boldsymbol{C} - \boldsymbol{P}$. Define also $\boldsymbol{\Delta}_{\text{ep}} = \boldsymbol{E} - \boldsymbol{P}$. Projectable points are classified according to the following list:

1. All ellipse points are projectable when $\boldsymbol{N} \cdot \boldsymbol{\Delta}_{\text{cp}} + r_{\max} < \boldsymbol{N} \cdot \boldsymbol{\Delta}_{\text{ep}}$.
2. None of the ellipse points are projectable when $\boldsymbol{N} \cdot \boldsymbol{\Delta}_{\text{cp}} - r_{\max} \geq \boldsymbol{N} \cdot \boldsymbol{\Delta}_{\text{ep}}$.
3. All ellipse points except for one point are projectable when $\boldsymbol{N} \cdot \boldsymbol{\Delta}_{\text{cp}} + r_{\max} \leq \boldsymbol{N} \cdot \boldsymbol{\Delta}_{\text{ep}}$. The exceptional point corresponds to $\hat{\theta}$ for which $r(\hat{\theta}) = r_{\max}$.
4. If the eye line and ellipse intersect in exactly 2 points, those points are not projectable. The ellipse minus the 2 points is a union of disjoint and open elliptical arcs. One arc has points that project onto the normal line in an interval $[\boldsymbol{N} \cdot \boldsymbol{\Delta}_{\text{cp}} - r_{\max}, \boldsymbol{N} \cdot \boldsymbol{\Delta}_{\text{ep}})$; these points are all projectable. The other arc has projection interval $(\boldsymbol{N} \cdot \boldsymbol{\Delta}_{\text{ep}}, \boldsymbol{N} \cdot \boldsymbol{\Delta}_{\text{cp}} + r_{\max}]$; none of these points are projectable.

The classification of the projection sets is described in the next section.

## 3 The Projection Algorithm

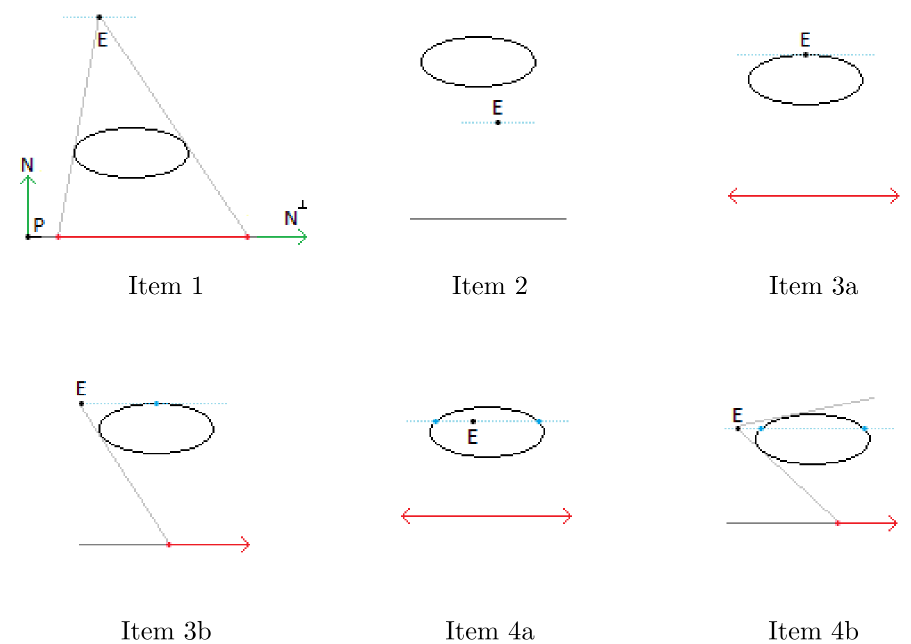
Projection sets are classified according to the following list.

1. All ellipse points are projectable when $\boldsymbol{N} \cdot \boldsymbol{\Delta}_{\text{cp}} + r_{\max} < \boldsymbol{N} \cdot \boldsymbol{\Delta}_{\text{ep}}$. The projection set is a segment.
2. None of the ellipse points are projectable when $\boldsymbol{N} \cdot \boldsymbol{\Delta}_{\text{cp}} - r_{\max} \geq \boldsymbol{N} \cdot \boldsymbol{\Delta}_{\text{ep}}$. The projection set is empty.
3. All ellipse points except for one point are projectable when $\boldsymbol{N} \cdot \boldsymbol{\Delta}_{\text{cp}} + r_{\max} \leq \boldsymbol{N} \cdot \boldsymbol{\Delta}_{\text{ep}}$. The exceptional point corresponds to $\hat{\theta}$ for which $\rho(\hat{\theta}) = r_{\max}$.
   (a) If $\boldsymbol{E}$ is on the ellipse, it is the exceptional point and the projection set is a line.
   (b) If $\boldsymbol{E}$ is not on the ellipse, the projection set is a ray.

4. If the eye line and ellipse intersect in exactly 2 points, those points are not projectable. The ellipse minus the 2 points is a union of disjoint and open elliptical arcs. One arc has points that project onto the normal line in an interval $[\boldsymbol{N} \cdot \boldsymbol{\Delta}_{\text{cp}} - r_{\max}, \boldsymbol{N} \cdot \boldsymbol{\Delta}_{\text{ep}})$; these points are all projectable. The other arc has projection interval $(\boldsymbol{N} \cdot \boldsymbol{\Delta}_{\text{ep}}, \boldsymbol{N} \cdot \boldsymbol{\Delta}_{\text{cp}} + r_{\max}]$; none of these points are projectable.
   (a) If $\boldsymbol{E}$ is strictly inside the ellipse, the projection set is a line.
   (b) If $\boldsymbol{E}$ is on or outside the ellipse, the projection set is a ray.

Figure 1 illustrates the items in the list.

**Figure 1.** The sets of projection for various geometric configurations. The labels correspond to the item numbering of the aforementioned list. The dotted blue line is the separator between projectable and unprojectable points. The red lines, rays and segment are projection sets.




Item 1, item 3b and item 4b require computing rays $\boldsymbol{R}(t) = \boldsymbol{E} + t\boldsymbol{D}$ that are tangent to the ellipse. The intersections of the rays with the projection line determine the segment endpoints (item 1) and the ray origins (items 3b and 4b). To determine the tangent directions $\boldsymbol{D}$, define $\boldsymbol{\Delta} = \boldsymbol{E} - \boldsymbol{C}$, substitute $\boldsymbol{X} = \boldsymbol{R}(t)$ into equation (2) and expand to obtain

$$0 = \left(\boldsymbol{D}^{\mathsf{T}} A \boldsymbol{D}\right) t^2 + 2\left(\boldsymbol{D}^{\mathsf{T}} A \boldsymbol{\Delta}\right) t + \left(\boldsymbol{\Delta}^{\mathsf{T}} A \boldsymbol{\Delta} - 1\right) = k_2 t^2 + 2k_1 t + k_0 \tag{6}$$

where the last equality defines the polynomial coefficients $k_0$, $k_1$ and $k_2$. The tangent directions are those for which the quadratic equation has a repeated real root $-k_1/k_2$, something that happens when

$$\begin{aligned} 0 &= k_1^2 - k_0 k_2 \\ &= \left(\boldsymbol{D}^\mathsf{T} A \boldsymbol{\Delta}\right)^2 - \left(\boldsymbol{D}^\mathsf{T} A \boldsymbol{D}\right)\left(\boldsymbol{\Delta}^\mathsf{T} A \boldsymbol{\Delta} - 1\right) \\ &= \boldsymbol{D}^\mathsf{T}\left[(A\boldsymbol{\Delta})(A\boldsymbol{\Delta})^\mathsf{T} + \left(1 - \boldsymbol{\Delta}^\mathsf{T} A \boldsymbol{\Delta}\right) A\right]\boldsymbol{D} \\ &= \boldsymbol{D}^\mathsf{T} B \boldsymbol{D} \end{aligned} \tag{7}$$

where the last equality defines the symmetric matrix $B$. Items 1, 3b and 4b in Figure 1 show that geometrically there must be 2 distinct tangent lines. Therefore, the matrix $B$ has 2 linearly independent eigenvectors. Moreover, equation (7) is possible only when $B$ is not positive definite and not negative definite; that is, the eigenvalues of $B$ have opposite signs or one of the eigenvalues is 0.

An eigendecomposition is $B = Q\Gamma Q^\mathsf{T}$, where $Q$ is an orthogonal matrix and $\Gamma = \mathrm{Diag}(\gamma_0, \gamma_1)$ with one of the conditions $\gamma_0 < 0 < \gamma_1$, $\gamma_0 = 0 < \gamma_1$ or $\gamma_0 < 0 = \gamma_1$. Defining $Q^\mathsf{T}\boldsymbol{D} = \boldsymbol{V} = [v_0\ v_1]^\mathsf{T}$, equation (7) becomes

$$0 = \left(Q^\mathsf{T}\boldsymbol{D}\right)^\mathsf{T} \Gamma \left(Q^\mathsf{T}\boldsymbol{D}\right) = \boldsymbol{V}^\mathsf{T}\Gamma\boldsymbol{V} = \gamma_0 v_0^2 + \gamma_1 v_1^2 \tag{8}$$

where also $v_0^2 + v_1^2 = 1$. We have 2 linear equations in the 2 unknowns $v_0^2$ and $v_1^2$. The 2 tangent directions are

$$\boldsymbol{D} = Q \left[\begin{array}{c} \pm\sqrt{\gamma_1/(\gamma_1 - \gamma_0)} \\ \sqrt{-\gamma_0/(\gamma_1 - \gamma_0)} \end{array}\right] \tag{9}$$

For item 1, both tangent rays are used to determine the endpoints of the segment of projection. For items 3b and 4b, only one of the tangent rays is used to determine the origin of the ray of projection. The ray to use is the one for which the tangent point $\boldsymbol{\tau}$ is projectable. The tangent points can be found by solving the quadratic polynomial in equation (6 for the unique real-valued root $\bar{t} = -\boldsymbol{D}^\mathsf{T} A\boldsymbol{\Delta}/\boldsymbol{D}^\mathsf{T} A \boldsymbol{D}_i$ computing $\boldsymbol{\tau} = \boldsymbol{R}(\bar{t}) = \boldsymbol{E} + \bar{t}\boldsymbol{D}$.

The intersection of the tangent ray $\boldsymbol{R}(s) = \boldsymbol{E} + s\boldsymbol{D}$ with the projection line is computed by substituting the ray equation into the equation for the project line, $\boldsymbol{N}\cdot(\boldsymbol{R}(s) - \boldsymbol{P}) = 0$ and solving for $\bar{s} = \boldsymbol{N}\cdot(\boldsymbol{P} - \boldsymbol{E})/\boldsymbol{N}\cdot\boldsymbol{D}$. The intersection point is $\boldsymbol{I} = \boldsymbol{E} + \bar{s}\boldsymbol{D}$.

## 4 Pseudocode for Computing the Tangent Rays and Intersections

Given an ellipse, an eye point strictly outside the ellipse and a projection line, Listing 1 contains pseudocode for computing the tangent rays and intersections with the projection line.

---

**Listing 1.** Pseudocode for computing the tangent rays and intersections. The eye point must be strictly outside the ellipse.

```
struct Ellipse2<T> { Vector2<T> C, U0, U1; T ell0, ell1; };
struct Line2<T> { Vector2<T> P, N; };

void TangentsAndIntersections(Vector2<T> eye, Line2<T> line, Ellipse2<T> ellipse,
    Vector2<T> tangent[2], Vector2<T> intersection[2])
{
```

```
    // Assert: The eye point must be strictly outside the ellipse.
    Matrix2x2<T> J = {{ellipse.U0[0], ellipse.U1[0]}, {ellipse.U0[1], ellipse.U1[1]}};
    Matrix2x2<T> L = {{ellipse.ell0, 0}, {0, ellipse.ell1}};
    Matrix2x2<T> invL = {{1 / L(0, 0), 0}, {0, 1 / L(1, 1)}};
    Matrix2x2<T> A = J * invL * invL * Transpose(J);
    Vector2<T> delta = eye - ellipse.C;
    Vector2<T> ADelta = A * delta;
    Matrix2x2<T> B = Adelta * Transpose(Adelta) + (1 - Dot(delta, ADelta)) * A;

    // The eigenvalues are ordered by γ0 < 0 < γ1 or γ0 = 0 < γ1 or γ0 < 0 = γ1.
    Matrix2x2<T> Q, Gamma;
    Eigendecomposition(B, Q, Gamma);

    Vector2<T> V;
    V[0] = sqrt(Gamma(1, 1) / (Gamma(1, 1) - Gamma(0, 0)));
    V[1] = -sqrt(-Gamma(0, 0) / (Gamma(1, 1) - Gamma(0, 0)));
    Vector2<T> D0 = Q * V;
    V[0] = -V[0];
    Vector2<T> D1 = Q * V;
    T t0 = -Dot(D0, ADelta) / Dot(D0, A * D0);
    T t1 = -Dot(D1, ADelta) / Dot(D1, A * D1);
    tangent[0] = eye + t0 * D0;
    tangent[1] = eye + t1 * D1;
    T s0 = Dot(line.N, projectionLine.P - eye) / Dot(line.N, D0);
    T s1 = Dot(line.N, projectionLine.P - eye) / Dot(line.N, D1);
    intersection[0] = eye + s0 * D0;
    intersection[1] = eye + s1 * D1;
}
```

# 5 Examples

The following examples were processed using Mathematica [Wol21]. The subsection titles refer to the item numbering in Section 3. The ellipse has center $\boldsymbol{C} = (0, 2)$, major axis direction $\boldsymbol{U}_0 = (\sqrt{3}, 1)/2$ with extent $d_0 = 2$ and minor axis direction $\boldsymbol{U}_1 = (-1, \sqrt{3})/2$ with extent $d_1 = 1/2$. The matrices of equation (2) are

$$J = \frac{1}{2}\begin{bmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{bmatrix}, \quad L = \begin{bmatrix} 2 & 0 \\ 0 & 1/2 \end{bmatrix}, \quad A = \frac{1}{16}\begin{bmatrix} 19 & -15\sqrt{3} \\ -15\sqrt{3} & 49 \end{bmatrix} \tag{10}$$

The projection line has origin $\boldsymbol{P} = (0, 0)$ and unit-length normal $\boldsymbol{N} = (0, 1)$ and unit-length direction $\boldsymbol{N}^{\perp} = (1, 0)$.

Mathematica can compute the intersections in closed form. However, in the following subsections, the relevant values are listed as floating-point approximations to their theoretical values using N[value].

## 5.1 The Configuration of Item 1

Let the eye point be $\boldsymbol{E} = (0, 5)$. The matrices of equation (7) are

$$B \doteq \begin{bmatrix} -7.8125 & -1.6238 \\ -1.6238 & 3.0625 \end{bmatrix}, \quad Q \doteq \begin{bmatrix} 0.989492 & -0.144591 \\ 0.144591 & 0.989492 \end{bmatrix}, \quad \Gamma \doteq \begin{bmatrix} -8.04978 & 0.0 \\ 0.0 & 3.29978 \end{bmatrix} \tag{11}$$

The two tangent directions are

$$\boldsymbol{D}_0 \doteq \begin{bmatrix} 0.655308 \\ -0.755361 \end{bmatrix}, \quad \boldsymbol{D}_1 \doteq \begin{bmatrix} -0.411767 \\ -0.911289 \end{bmatrix} \tag{12}$$

The tangent points occur at $t_0 \doteq 2.62161$ and $t_1 \doteq 4.17217$. The points themselves are

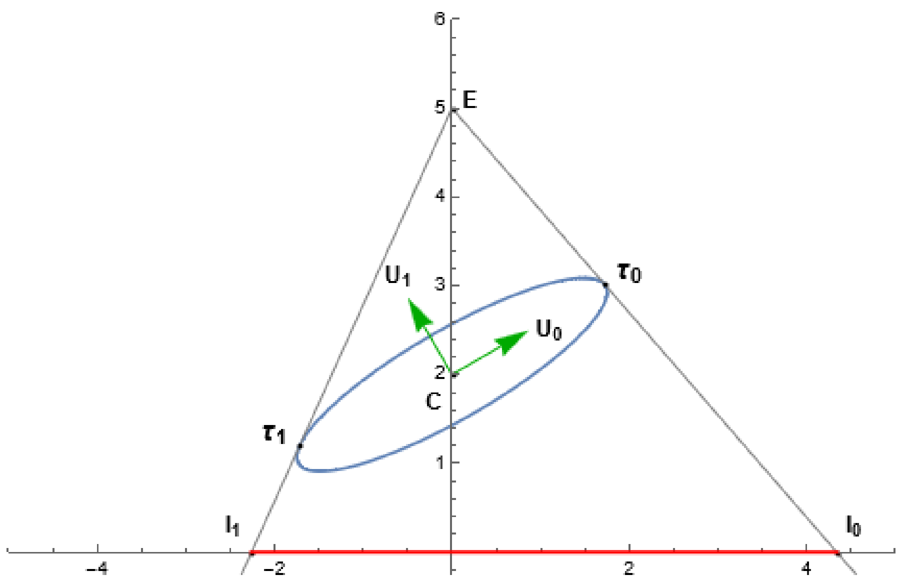
$$\boldsymbol{\tau}_0 \doteq \begin{bmatrix} 1.71796 \\ 3.01974 \end{bmatrix}, \quad \boldsymbol{\tau}_1 \doteq \begin{bmatrix} -1.71796 \\ 1.19795 \end{bmatrix} \tag{13}$$

The intersection points occur at $s_0 \doteq 6.61935$ and $s_1 == \doteq 5.48673$. The points themselves are

$$\boldsymbol{I}_0 \doteq \begin{bmatrix} 4.33771 \\ 0.0 \end{bmatrix}, \quad \boldsymbol{I}_1 \doteq \begin{bmatrix} -2.25925 \\ 0.0 \end{bmatrix} \tag{14}$$

Figure 2 shows an ellipse drawn in blue. The ellipse axis directions are drawn in green. The tangent rays are drawn in gray. The projection set is a segment drawn in red.

**Figure 2.** An illustration of the geometric configuration of Item 1 where the projection is a segment.




## 5.2 The Configuration of Item 3b

The ellipse point that attains maximum $\boldsymbol{N} \cdot (\boldsymbol{X} - \boldsymbol{C})$ is $\boldsymbol{X} = \boldsymbol{C} + A^{-1}\boldsymbol{N}/\sqrt{\boldsymbol{N}^{\mathsf{T}} A^{-1} \boldsymbol{N}}$. At this point, the normal vector to the ellipse is in the direction $\boldsymbol{N}$. Let the eye point be

$$\boldsymbol{E} = \boldsymbol{C} + A^{-1}\boldsymbol{N}/\sqrt{\boldsymbol{N}^{\mathsf{T}} A^{-1} \boldsymbol{N}} - 4\boldsymbol{N}^{\perp} = \left(-4 + (3)^{3/4}\sqrt{5}/4, 2 + \sqrt{19}/4\right) \doteq (-2.72572, 3.08972) \tag{15}$$

The matrices of equation (7) are

$$B \doteq \begin{bmatrix} 0.0 & -4.59408 \\ -4.59408 & -4.36703 \end{bmatrix}, \quad Q \doteq \begin{bmatrix} 0.534196 & -0.845361 \\ 0.845361 & 0.534196 \end{bmatrix}, \quad \Gamma \doteq \begin{bmatrix} -7.2701 & 0.0 \\ 0.0 & 2.90306 \end{bmatrix} \tag{16}$$

The two tangent directions are

$$\boldsymbol{D}_0 \doteq \begin{bmatrix} 1.0 \\ 0.0 \end{bmatrix}, \quad \boldsymbol{D}_1 \doteq \begin{bmatrix} 0.42927 \\ -0.903176 \end{bmatrix} \tag{17}$$

The tangent points occur at $t_0 \doteq 4.21582$ and $t_1 \doteq 2.30393$. The points themselves are

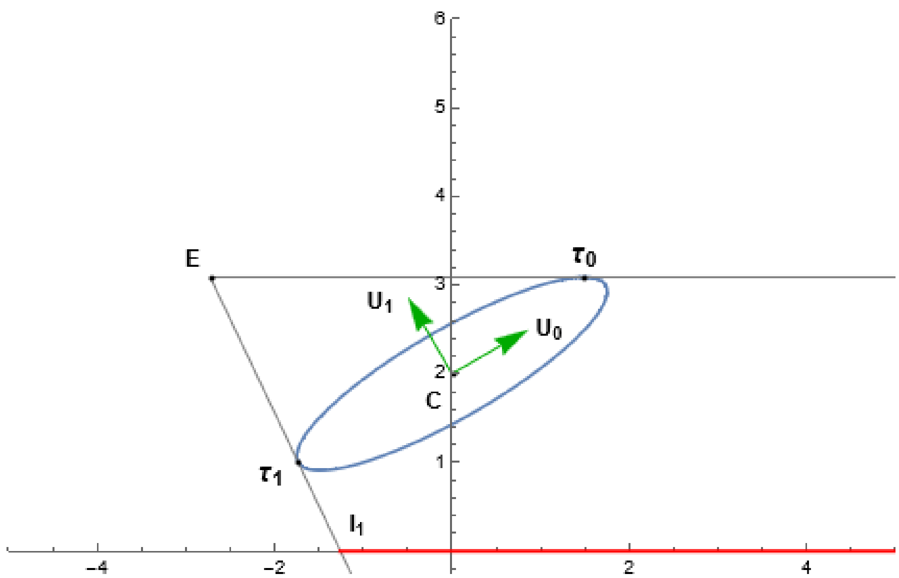
$$\boldsymbol{\tau}_0 \doteq \begin{bmatrix} 1.4901 \\ 3.08972 \end{bmatrix}, \quad \boldsymbol{\tau}_1 \doteq \begin{bmatrix} -1.73671 \\ 1.00887 \end{bmatrix} \tag{18}$$

The intersection point occurs at $s_1 \doteq 3.42095$. The point itself is

$$\boldsymbol{I}_1 \doteq \begin{bmatrix} -1.2572 \\ 0.0 \end{bmatrix} \tag{19}$$

Figure 3 shows an ellipse drawn in blue. The ellipse axis directions are drawn in green. The tangent rays are drawn in gray. The projection set is a ray drawn in red.

**Figure 3.** An illustration of the geometric configuration of Item 3b where the projection is a ray.



### 5.3 The Configuration of Item 4b

The ellipse point that attains maximum $\boldsymbol{N} \cdot (\boldsymbol{X} - \boldsymbol{C})$ is $\boldsymbol{X} = \boldsymbol{C} + A^{-1}\boldsymbol{N}/\sqrt{\boldsymbol{N}^\mathsf{T} A^{-1}\boldsymbol{N}}$. At this point, the normal vector to the ellipse is in the direction $\boldsymbol{N}$.

Let the eye point be

$$\boldsymbol{E} = \boldsymbol{C} + A^{-1}\boldsymbol{N}/\sqrt{\boldsymbol{N}^\mathsf{T} A^{-1}\boldsymbol{N}} - 4\boldsymbol{N}^\perp - \boldsymbol{N} = \left(-4 + (3)^{3/4}\sqrt{5}/4, \sqrt{19}/4\right) \doteq (-2.72572, 2.08972) \tag{20}$$

The matrices of equation (7) are

$$B \doteq \begin{bmatrix} 1.17945 & -1.86836 \\ -1.86836 & -4.36703 \end{bmatrix}, \quad Q \doteq \begin{bmatrix} 0.292109 & -0.956385 \\ 0.956385 & 0.292109 \end{bmatrix}, \quad \Gamma \doteq \begin{bmatrix} -4.93769 & 0.0 \\ 0.0 & 1.7501 \end{bmatrix} \tag{21}$$

The two tangent directions are

$$\boldsymbol{D}_0 \doteq \begin{bmatrix} 0.971205 \\ 0.238246 \end{bmatrix}, \quad \boldsymbol{D}_1 \doteq \begin{bmatrix} 0.672346 \\ -0.740237 \end{bmatrix} \tag{22}$$

The tangent points occur at $t_0 \doteq 3.99119$ and $t_1 \doteq 1.50185$. The points themselves are

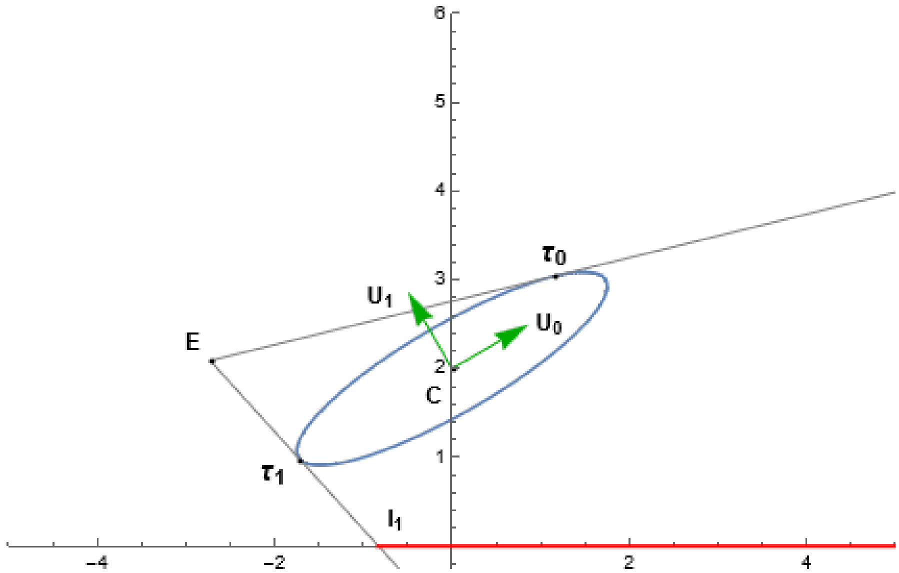
$$\boldsymbol{\tau}_0 \doteq \begin{bmatrix} 1.15054 \\ 3.04061 \end{bmatrix}, \quad \boldsymbol{\tau}_1 \doteq \begin{bmatrix} -1.71596 \\ 0.978003 \end{bmatrix} \tag{23}$$

The intersection point occurs at $s_1 \doteq 2.82305$. The point itself is

$$\boldsymbol{I}_1 \doteq \begin{bmatrix} -0.827652 \\ 0.0 \end{bmatrix} \tag{24}$$

Figure 4 shows an ellipse drawn in blue. The ellipse axis directions are drawn in green. The tangent rays are drawn in gray. The projection set is a ray drawn in red.

**Figure 4.** An illustration of the geometric configuration of Item 4b where the projection is a ray.




# References


[Wol21] Wolfram Research, Inc. *Mathematica 12.3.1.* Wolfram Research, Inc., Champaign, Illinois, 2021.