Perspective Projection of an Ellipse

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1 Introduction

The standard axis-aligned ellipse in 2D is of the form

$$\left(\frac{y_0}{d_0}\right)^2 + \left(\frac{y_1}{d_1}\right)^2 = 1 \tag{1}$$

where $d_0 > 0$ and $d_1 > 0$. The 2D coordinates of points are (y_0, y_1).

In 3D, we can specify an ellipse in a general plane. Let the ellipse center be $\mathbf{C}_e$, which is a point on the plane. Let the plane normal be the unit-length vector $\mathbf{N}_e$. Choose unit-length vectors $\mathbf{U}_e$ and $\mathbf{V}_e$ so that the set $\{\mathbf{U}_e, \mathbf{V}_e, \mathbf{N}_e\}$ is a right-handed orthonormal basis; that is, the vectors are unit-length, mutually orthogonal, and $\mathbf{N}_e = \mathbf{U}_e \times \mathbf{V}_e$. The ellipse consists of 3D points

$$\mathbf{X}_e = \mathbf{C}_e + y_0\mathbf{U}_e + y_1\mathbf{V}_e = \mathbf{C}_e + J_e\mathbf{Y}_e \tag{2}$$

where (y_0, y_1) is a solution to equation (1). The second equality in equation (2) defines the 3×2 matrix J_e whose columns are $\mathbf{U}_e$ and $\mathbf{V}_e$ and defines the 2×1 column vector $\mathbf{Y}_e$ whose rows are the components of (y_0, y_1).

Given an eyepoint $\mathbf{E}$, we want to perspectively project the ellipse onto another plane. That plane has unit-length normal $\mathbf{N}_p$ and origin point $\mathbf{C}_p$. We may choose the normal so that $\mathbf{E}$ is on the side of the plane to which $\mathbf{N}_p$ is directed; that is, $\mathbf{N}_p \cdot (\mathbf{E} - \mathbf{C}_p) > 0$. If $\{\mathbf{U}_p, \mathbf{V}_p, \mathbf{N}_p\}$ is a right-handed orthonormal set, then points on the projection plane are

$$\mathbf{X}_p = \mathbf{C}_p + y_2\mathbf{U}_p + y_3\mathbf{V}_p = \mathbf{C}_p + J_p\mathbf{Y}_p \tag{3}$$

The second equality in equation (3) defines the 3×2 matrix J_p whose columns are $\mathbf{U}_p$ and $\mathbf{V}_p$ and defines the 2×1 column vector $\mathbf{Y}_p$ whose rows are the components of (y_2, y_3). Assuming the ellipse is between the eyepoint and the projection plane, the projection is also an ellipse. We will determine a quadratic equation in the components of $\mathbf{Y}_p$ that define the projected ellipse.

2 Elliptical Cones

There are multiple ways to approach the projection problem. The one presented here is based on computing an ellipse as the intersection of an elliptical cone and a plane. The intersection will be represented implicitly by a quadratic equation of $\mathbf{Y}_p$.

2.1 Right Elliptical Cones

A standard right elliptical cone is represented algebraically as

$$\left(\frac{y_2}{e_2} - 1\right)^2 = \left(\frac{y_0}{d_0}\right)^2 + \left(\frac{y_1}{d_1}\right)^2 \tag{4}$$

where the cone vertex is located at $(0, 0, e_2)$ with $e_2 > 0$. The cone vertex is a solution to equation (4). When $y_2 = 0$, the elliptical cone equation reduces to the standard ellipse equation (1). Thus, the standard ellipse points are on the elliptical cone, which means that these points are the intersection of the elliptical cone with the y_0y_1-plane.

2.2 Skewed Elliptical Cones

The use of the term right elliptical refers to the vector from ellipse origin to cone vertex is perpendicular to the ellipse plane. We can define an elliptical cone generally by allowing the cone vertex to be at a point $\bar{\mathbf{E}} = (\bar{e}_0, \bar{e}_1, \bar{e}_2)$ with $\bar{e}_2 > 0$ and the first two components not necessarily zero. The cone axis passes through $\bar{\mathbf{E}}$ and the origin $(0, 0, 0)$ but is no longer perpendicular to the ellipse plane. We wish to have an algebraic representation of this cone. Let us first look at a derivation that leads to equation (4).

Define $Q(\mathbf{Y}) = \mathbf{Y}^{\mathsf{T}}\bar{A}\mathbf{Y} + \bar{\mathbf{B}}^{\mathsf{T}}\mathbf{Y} + \bar{c}$, where as a 3-tuple, $\mathbf{Y} = (y_0, y_1, y_2)$. The 3×3 matrix $\bar{A}$ is symmetric, $\bar{\mathbf{B}}$ is a 3×1 vector, and $\bar{c}$ is a scalar. The right elliptical cone is represented by the quadratic equation $Q(\mathbf{Y}) = 0$ for some choice of $\bar{A}$, $\bar{\mathbf{B}}$, and $\bar{c}$. Observe that the choices involve 10 parameters, 6 for $\bar{A} = [\bar{a}_{ij}]$, 3 for $\bar{\mathbf{B}} = [\bar{b}_j]$, and 1 for $\bar{c}$.

1. When $y_2 = 0$, $Q(\mathbf{Y}) = 0$ must reduce to the standard ellipse equation. Replacing $y_2 = 0$ in the quadratic equation,

$$\begin{bmatrix} y_0 & y_1 \end{bmatrix} \begin{bmatrix} \bar{a}_{00} & \bar{a}_{01} \\ \bar{a}_{01} & \bar{a}_{11} \end{bmatrix} \begin{bmatrix} y_0 \\ y_1 \end{bmatrix} + \begin{bmatrix} \bar{b}_0 & \bar{b}_1 \end{bmatrix} \begin{bmatrix} y_0 \\ y_1 \end{bmatrix} + \bar{c} = 0$$

   To obtain the standard ellipse equation, we need $\bar{a}_{00} = 1/d_0^2$, $\bar{a}_{01} = 0$, $\bar{a}_{11} = 1/d_1^2$, $\bar{b}_0 = 0$, $\bar{b}_1 = 0$, and $\bar{c} = -1$.

2. The surface normal at the cone vertex is degenerate, so the gradient must be the zero vector: $\mathbf{0} = \nabla Q(\bar{\mathbf{E}}) = 2\bar{A}\bar{\mathbf{E}} + \bar{\mathbf{B}}$. Using the information from item 1 and this constraint, we have $\bar{a}_{02} = -\bar{e}_0/(\bar{e}_2 d_0^2)$, $\bar{a}_{12} = -\bar{e}_1/(\bar{e}_2 d_1^2)$, and $2\bar{e}_2^2\bar{a}_{22} + \bar{e}_2\bar{b}_2 = 2(\bar{e}_0^2/d_0^2 + \bar{e}_1^2/d_1^2)$.

3. The cone vertex solves $Q(\bar{\mathbf{E}}) = 0$, which implies $\bar{e}_2^2\bar{a}_{22} + \bar{e}_2\bar{b}_2 = 1 + (\bar{e}_0^2/d_0^2 + \bar{e}_1^2/d_1^2)$.

4. Items 2 and 3 are solved for $\bar{a}_{22} = (\bar{e}_0^2/d_0^2 + \bar{e}_1^2/d_1^2 - 1)/\bar{e}_2^2$ and $\bar{b}_2 = 2/\bar{e}_2$.

We have determined A, $\mathbf{B}$, and c, namely,

$$\bar{A} = \begin{bmatrix} 1/d_0^2 & 0 & -\bar{e}_0/(\bar{e}_2 d_0^2) \\ 0 & 1/d_1^2 & -\bar{e}_1/(\bar{e}_2 d_1^2) \\ -\bar{e}_0/(\bar{e}_2 d_0^2) & -\bar{e}_1/(\bar{e}_2 d_1^2) & (\bar{e}_0^2/d_0^2 + \bar{e}_1^2/d_1^2 - 1)/\bar{e}_2^2 \end{bmatrix}, \quad \bar{\mathbf{B}} = \begin{bmatrix} 0 \\ 0 \\ 2/\bar{e}_2 \end{bmatrix}, \quad \bar{c} = -1 \tag{5}$$

Some factorization leads to the equation for the elliptical cone,

$$\left(\frac{y_2}{\bar{e}_2} - 1\right)^2 = \left(\frac{y_0}{d_0} - \frac{\bar{e}_0}{d_0}\frac{y_2}{\bar{e}_2}\right)^2 + \left(\frac{y_1}{d_1} - \frac{\bar{e}_1}{d_1}\frac{y_2}{\bar{e}_2}\right)^2 \tag{6}$$

2.3 General Elliptical Cones

In the most general case, the ellipse is not necessary in a coordinate-axis plane. Equation (2) is the equation for ellipse points, where (y_0, y_1) solves equation (1). We want to determine A, $\mathbf{B}$, and c for which $Q(\mathbf{X}) = 0$ implicitly defines the elliptical cone with vertex $\mathbf{E}$ and contains the specified ellipse points $\mathbf{X}_e$.

First, the ellipse points must satisfy $0 = Q(\mathbf{X}) = \mathbf{X}^\mathsf{T} A\mathbf{X} + \mathbf{B}^\mathsf{T}\mathbf{X} + c$,

$$\begin{aligned} 0 &= \mathbf{X}_e^\mathsf{T} A\mathbf{X}_e + \mathbf{B}^\mathsf{T}\mathbf{X}_e + c \\ &= (\mathbf{C}_e + J_e\mathbf{Y}_e)^\mathsf{T} A\,(\mathbf{C}_e + J_e\mathbf{Y}_e) + \mathbf{B}^\mathsf{T}(\mathbf{C}_e + J_e\mathbf{Y}_e) + c \\ &= \mathbf{Y}_e^\mathsf{T}\left(J_e^\mathsf{T} A J_e\right)\mathbf{Y}_e + \left(J_e^\mathsf{T}(\mathbf{B} + 2A\mathbf{C}_e)\right)^\mathsf{T}\mathbf{Y}_e + \left(\mathbf{C}_e^\mathsf{T} A\mathbf{C}_e + \mathbf{B}^\mathsf{T}\mathbf{C}_e + c\right) \end{aligned} \tag{7}$$

Because $\mathbf{Y}_e$ is a solution to the standard ellipse equation,

$$J_e^\mathsf{T} A J_e = \mathrm{Diagonal}(1/d_0^2, 1/d_1^2), \quad J_e^\mathsf{T}(\mathbf{B} + 2A\mathbf{C}_e) = (0,0), \quad \mathbf{C}_e^\mathsf{T} A\mathbf{C}_e + \mathbf{B}^\mathsf{T}\mathbf{C}_e + c = -1 \tag{8}$$

which is a set of 6 linear equations in the unknowns. Second, the gradient of Q must be the zero vector at the cone vertex,

$$\nabla Q(\mathbf{E}) = 2A\mathbf{E} + \mathbf{B} = \mathbf{0} \tag{9}$$

which is a set of 3 linear equations in the unknowns. Third, the cone vertex is a solution to the quadratic equation,

$$Q(\mathbf{E}) = \mathbf{E}^\mathsf{T} A\mathbf{E} + \mathbf{B}^\mathsf{T}\mathbf{E} + c = 0 \tag{10}$$

which is 1 linear equation in the unknowns. Altogether we have 10 linear equations in the 10 unknowns, allowing us to use a linear system solver to obtain the solution.

2.4 Converting General Cones to Skewed Cones

Alternatively, we may translate the ellipse center to the origin and then rotate the ellipse plane into a coordinate plane, thus transforming the problem to the one we know has the representation in equation (6). Let $R_e = [\mathbf{U}_e\ \mathbf{V}_e\ \mathbf{N}_e]$ be the rotation matrix whose columns are the right-handed orthonormal basis mentioned in the introduction. A point $\mathbf{X}$ in the current coordinate system is related to a point $\mathbf{Y}$ in the coordinate system of the skewed cone via the transformations

$$\mathbf{X} = \mathbf{C}_e + R_e\mathbf{Y}, \quad \mathbf{Y} = R_e^\mathsf{T}(\mathbf{X} - \mathbf{C}_e) \tag{11}$$

The eyepoint $\mathbf{E}$ is transformed to an eyepoint $\bar{\mathbf{E}} = R_e^\mathsf{T}(\mathbf{E} - \mathbf{C}_e) = (\bar{e}_0, \bar{e}_1, \bar{e}_2)$. Equation (6) provides the ellipse representation

$$\left(\frac{y_2}{\bar{e}_2} - 1\right)^2 = \left(\frac{y_0}{d_0} - \frac{\bar{e}_0}{d_0}\frac{y_2}{\bar{e}_2}\right)^2 + \left(\frac{y_1}{d_1} - \frac{\bar{e}_1}{d_1}\frac{y_2}{\bar{e}_2}\right)^2 \tag{12}$$

In summary, the vertex of the general elliptical cone is $\mathbf{E}$. The ellipse consists of points $\mathbf{X}_e = \mathbf{C}_e + y_0\mathbf{U}_e + y_1\mathbf{V}_e$, where (y_0, y_1) is a solution to equation (12) with $\bar{\mathbf{E}} = R_e^\mathsf{T}(\mathbf{E} - \mathbf{C}_e)$ and $R_e = [\mathbf{U}_e\ \mathbf{V}_e\ \mathbf{N}_e]$.

For later use, we require the quadratic equation in the transformed coordinate system,

$$\begin{aligned} 0 &= \mathbf{X}^\mathsf{T} A\mathbf{X} + \mathbf{B}^\mathsf{T}\mathbf{X} + c \\ &= (\mathbf{C}_e + R_e\mathbf{Y})^\mathsf{T} A\,(\mathbf{C}_e + R_e\mathbf{Y}) + \mathbf{B}^\mathsf{T}(\mathbf{C}_e + R_e\mathbf{Y}) + c \\ &= \mathbf{Y}^\mathsf{T}\left(R_e^\mathsf{T} A R_e\right)\mathbf{Y} + \left(R_e^\mathsf{T}(\mathbf{B} + 2A\mathbf{C}_e)\right)^\mathsf{T}\mathbf{Y} + \left(\mathbf{C}_e^\mathsf{T} A\mathbf{C}_e + \mathbf{B}^\mathsf{T}\mathbf{C}_e + c\right) \\ &= \mathbf{Y}^\mathsf{T}\bar{A}\mathbf{Y} + \bar{\mathbf{B}}^\mathsf{T}\mathbf{Y} + \bar{c} \end{aligned} \tag{13}$$

where the last equality defines the 3×3 matrix $\bar{A}$, the 3×1 vector $\bar{\mathbf{B}}$, and the scalar $\bar{c}$; that is,

$$\bar{A} = R_e^{\mathsf{T}} A R_e, \quad \bar{\mathbf{B}} = R_e^{\mathsf{T}}(\mathbf{B} + 2A\mathbf{C}_e), \quad \bar{c} = \mathbf{C}_e^{\mathsf{T}} A \mathbf{C}_e + \mathbf{B}^{\mathsf{T}}\mathbf{C}_e + c \tag{14}$$

where $\bar{A}$, $\bar{\mathbf{B}}$, and $\bar{c}$ are constructued using equation (5). This determines the coefficients of the $\mathbf{X}$-quadratic; define $\bar{\mathbf{C}}_e = R_e^{\mathsf{T}}\mathbf{C}_e$; then

$$A = R_e \bar{A} R_e^{\mathsf{T}}, \quad \mathbf{B} = R_e(\bar{\mathbf{B}} - 2\bar{A}\bar{\mathbf{C}}_e), \quad c = \bar{\mathbf{C}}_e^{\mathsf{T}} \bar{A} \bar{\mathbf{C}}_e - \bar{\mathbf{B}}^{\mathsf{T}}\bar{\mathbf{C}}_e + \bar{c} \tag{15}$$

3 Projection of the Ellipse to Another Plane

We now have a representation of the general elliptical cone that contains the eyepoint $\mathbf{E}$ and the user-specified ellipse: $\mathbf{X}^{\mathsf{T}} A \mathbf{X} + \mathbf{B}^{\mathsf{T}}\mathbf{X} + c = 0$, where A, $\mathbf{B}$, and c are defined by equation (15). The quantities $\bar{A}$, $\bar{\mathbf{B}}$, and $\bar{c}$ are determined using $\bar{\mathbf{E}} = R_e^{\mathsf{T}}(\mathbf{E} - \mathbf{C}_e)$ in the skewed ellipse equation (5).

We have the projection plane, whose origin is the point $\mathbf{C}_p$ and unit-length normal is $\mathbf{N}_p$, and a right-handed orthonormal basis $\{\mathbf{U}_p, \mathbf{V}_p, \mathbf{N}_p\}$ that may be used to represent points relative to this plane. Assume that this projection plane intersects the cone in a closed curve, which is necessarily an ellipse. The plane points are specified by equation (3). To compute the intersection of cone and plane, substitute the plane points into the quadratic equation for the elliptical cone,

$$\begin{aligned} 0 &= \mathbf{X}_p^{\mathsf{T}} A \mathbf{X}_p + \mathbf{B}^{\mathsf{T}}\mathbf{X}_p + c \\ &= (\mathbf{C}_p + J_p \mathbf{Y}_p)^{\mathsf{T}} A (\mathbf{C}_p + J_p \mathbf{Y}_p) + \mathbf{B}^{\mathsf{T}} (\mathbf{C}_p + J_p \mathbf{Y}_p) + c \\ &= \mathbf{Y}_p^{\mathsf{T}} \left(J_p^{\mathsf{T}} A J_p\right) \mathbf{Y}_p + \left(J_p^{\mathsf{T}}(\mathbf{B} + 2A\mathbf{C}_p)\right)^{\mathsf{T}} \mathbf{Y}_p + \left(\mathbf{C}_p^{\mathsf{T}} A \mathbf{C}_p + \mathbf{B}^{\mathsf{T}}\mathbf{C}_p + c\right) \\ &= \mathbf{Y}_p^{\mathsf{T}} \hat{A} \mathbf{Y}_p + \hat{\mathbf{B}}^{\mathsf{T}}\mathbf{Y}_p + \hat{c} \end{aligned} \tag{16}$$

where the last equality defines $\hat{A}$, $\hat{\mathbf{B}}$, and $\hat{c}$; that is,

$$\hat{A} = J_p^{\mathsf{T}} A J_p, \quad \hat{\mathbf{B}} = J_p^{\mathsf{T}}(\mathbf{B} + 2A\mathbf{C}_p), \quad \hat{c} = \mathbf{C}_p^{\mathsf{T}} A \mathbf{C}_p + \mathbf{B}^{\mathsf{T}}\mathbf{C}_p + c \tag{17}$$

The equation for the projected ellipse may be factored into

$$(\mathbf{Y}_p - \mathbf{K})^{\mathsf{T}} M (\mathbf{Y}_p - \mathbf{K}) = 1, \tag{18}$$

where $\mathbf{K} = -\hat{A}^{-1}\hat{\mathbf{B}}/2$ and $M = \hat{A}/(\hat{\mathbf{B}}^{\mathsf{T}}\hat{A}^{-1}\hat{\mathbf{B}}/4 - \hat{c})$. To obtain axis vectors and extents, you may apply an eigendecomposition of $M = RDR^{\mathsf{T}}$. The columns of R are the axis vectors and the reciprocals of the diagonal entries of D are the squared extents.

The outline of the algorithm is listed next.

1. **Inputs**:
   (a) Eyepoint $\mathbf{E}$.
   (b) Ellipse plane containing ellipse center $\mathbf{C}_e$, unit-length normal $\mathbf{N}_e$, and unit-length ellipse axes $\mathbf{U}_e$ and $\mathbf{V}_e$. The matrix $R_e = [\mathbf{U}_e\, \mathbf{V}_e\, \mathbf{N}_e]$ is a rotation matrix. Ellipse axis lengths d_0 (associated with $\mathbf{U}_e$) and d_1 (associated with $\mathbf{V}_e$). Define $J_e = [\mathbf{U}_e\ \mathbf{V}_e]$. The ellipse points are $\mathbf{X}_e = \mathbf{C}_e + J_e\mathbf{Y}_e$ with $\mathbf{Y}_e^{\mathsf{T}}\text{Diagonal}(1/d_0^2, 1/d_1^2)\mathbf{Y}_e = 1$.
   (c) Projection plane with origin $\mathbf{C}_p$, unit-length normal $\mathbf{N}_p$, and vectors $\mathbf{U}_p$ and $\mathbf{V}_p$ that span the plane. The matrix $R_p = [\mathbf{U}_p\ \mathbf{V}_p\ \mathbf{N}_p]$ is a rotation matrix. Define $J_p = [\mathbf{U}_p\ \mathbf{V}_p]$. The projected ellipse points are $\mathbf{X}_p = \mathbf{C}_p + J_p\mathbf{Y}_p$, and the goal is to determine a quadratic equation in $\mathbf{Y}_p$ that represents this ellipse.

2. **Compute**:
   (a) $\bar{\mathbf{E}} = R_e^{\mathsf{T}}(\mathbf{E} - \mathbf{C}_e) = (\bar{e}_0, \bar{e}_1, \bar{e}_2)$, $\bar{\mathbf{C}}_e = R_e^{\mathsf{T}}\mathbf{C}_e$.
   (b) From equation (5) compute $\bar{A} = [\bar{a}_{ij}]$, $\bar{\mathbf{B}} = [\bar{b}_j]$, and $\bar{c}$:

   $$\bar{a}_{00} = 1/d_0^2,\ \ \bar{a}_{11} = 1/d_1^2,\ \ \bar{a}_{01} = \bar{a}_{10} = 0,\ \ \bar{a}_{02} = \bar{a}_{20} = -\bar{e}_0/(\bar{e}_2 d_0^2),\ \ \bar{a}_{12} = \bar{a}_{21} = -\bar{e}_1/(\bar{e}_2 d_1^2),$$
   $$\bar{a}_{22} = (\bar{e}_0^2/d_0^2 + \bar{e}_1^2/d_1^2 - 1)/\bar{e}_2^2,\ \ \bar{b}_0 = 0,\ \ \bar{b}_1 = 0,\ \ \bar{b}_2 = 2/\bar{e}_2,\ \ \bar{c} = -1$$

   (c) From equation (15) compute A, $\mathbf{B}$, and c:

   $$A = R_e\bar{A}R_e^{\mathsf{T}},\ \ \mathbf{B} = R_e(\bar{\mathbf{B}} - 2\bar{A}\bar{\mathbf{C}}_e),\ \ c = \bar{\mathbf{C}}_e^{\mathsf{T}}\bar{A}\bar{\mathbf{C}}_e - \bar{\mathbf{B}}^{\mathsf{T}}\bar{\mathbf{C}}_e + \bar{c}$$

   (d) From equation (17) compute $\hat{A}$, $\hat{\mathbf{B}}$, and $\hat{c}$:

   $$\hat{A} = J_p^{\mathsf{T}}AJ_p,\ \ \hat{\mathbf{B}} = J_p^{\mathsf{T}}(\mathbf{B} + 2A\mathbf{C}_p),\ \ \hat{c} = \mathbf{C}_p^{\mathsf{T}}A\mathbf{C}_p + \mathbf{B}^{\mathsf{T}}\mathbf{C}_p + c$$

3. **Outputs**:
   (a) The quadratic equation

   $$\mathbf{Y}_p^{\mathsf{T}}\hat{A}\mathbf{Y}_p + \hat{\mathbf{B}}\mathbf{Y}_p + \hat{c}$$

   that defines the projected ellipse.
   (b) The factored equation

   $$(\mathbf{Y}_p - \mathbf{K})^{\mathsf{T}} M\, (\mathbf{Y}_p - \mathbf{K}) = 1,$$

   where $\mathbf{K} = -\hat{A}^{-1}\hat{\mathbf{B}}/2$ and $M = \hat{A}/(\hat{\mathbf{B}}^{\mathsf{T}}\hat{A}^{-1}\hat{\mathbf{B}}/4 - \hat{c})$.
   (c) Eigendecomposition $RDR^{\mathsf{T}} = M$, where the columns of R are the projected ellipse axis vectors and $D = \text{Diagonal}(1/\lambda_0^2, 1/\lambda_1^2)$. The projected ellipse axis lengths are λ_0 and λ_1.