Perspective Mappings

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Contents

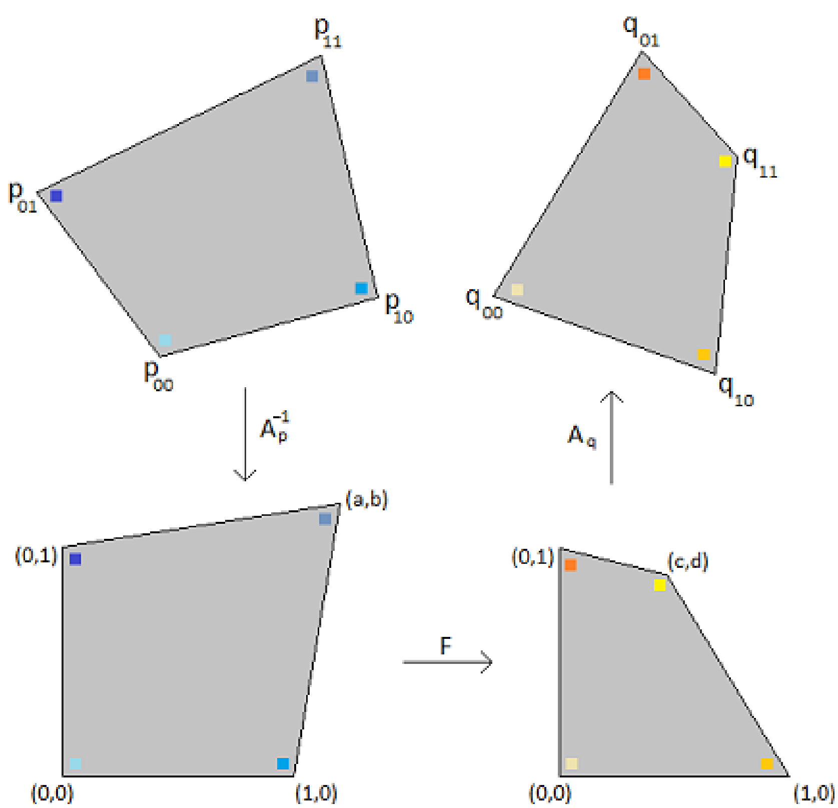
1 Introduction

This document describes an algorithm for perspective mappings between two convex quadrilaterals in 2D. In 3D, a cuboid is a convex polyhedron with 6 convex quadrilateral faces. It is not possible to extend the 2D algorithm to all pairs of cuboids in 3D. However, there are subsets of cuboids for which perspective mappings can be constructed. (The versions of this document through 01April2019 were in error regarding existence of perspective mappings between two arbitrary cuboids. In particular, the versions were also in error about the existence of what I called *parallax projections*. This document corrects those errors.)

2 Mapping Between Convex Quadrilaterals

The first convex quadrilateral has vertices $\boldsymbol{p}_{00}$, $\boldsymbol{p}_{10}$, $\boldsymbol{p}_{11}$, and $\boldsymbol{p}_{01}$, listed in counterclockwise order. The second convex quadrilateral has vertices $\boldsymbol{q}_{00}$, $\boldsymbol{q}_{10}$, $\boldsymbol{q}_{11}$, and $\boldsymbol{q}_{01}$, listed in counterclockwise order. The construction of the perspective mapping uses 3×3 homogeneous matrices and 3×1 homogeneous coordinates. Figure 1 shows the two convex quadrilaterals.

Figure 1. Two convex quadrilaterals. The upper-left quadrilateral is to be mapped perspectively to the upper-right quadrilateral. The transforms A_p and A_q are affine and F is a fractional linear transformation.



Affinely transform the source quadrilateral so that $\boldsymbol{p}_{00}$ is mapped to the origin $(0,0)$, $\boldsymbol{p}_{10} - \boldsymbol{p}_{00}$ is mapped to $(1,0)$ and $\boldsymbol{p}_{01} - \boldsymbol{p}_{00}$ is mapped to $(0,1)$. This is equivalent to computing $\boldsymbol{x} = (x_0, x_1)$ for which a point $\boldsymbol{p}$ in the upper-left quadrilateral is represented by

$$\boldsymbol{p} = \boldsymbol{p}_{00} + x_0(\boldsymbol{p}_{10} - \boldsymbol{p}_{00}) + x_1(\boldsymbol{p}_{01} - \boldsymbol{p}_{00}) \tag{1}$$

The representation of $\boldsymbol{p}_{11}$ leads to $(x_0, x_1) = (a, b)$, which is shown in the lower-left quadrilateral of figure 1. In homogeneous coordinates we have

$$\left[\begin{array}{c} \boldsymbol{p} \\ \hline 1 \end{array}\right] = \left[\begin{array}{cc|c} \boldsymbol{p}_{10} - \boldsymbol{p}_{00} & \boldsymbol{p}_{01} - \boldsymbol{p}_{00} & \boldsymbol{p}_{00} \\ \hline 0 & 0 & 1 \end{array}\right] \left[\begin{array}{c} \boldsymbol{x} \\ \hline 1 \end{array}\right] = \left[\begin{array}{c|c} M_p & \boldsymbol{p}_{00} \\ \hline \mathbf{0}^\mathsf{T} & 1 \end{array}\right] \left[\begin{array}{c} \boldsymbol{x} \\ \hline 1 \end{array}\right] = A_p \left[\begin{array}{c} \boldsymbol{x} \\ \hline 1 \end{array}\right] \tag{2}$$

where M_p is a 2×2 matrix whose columns are $\boldsymbol{p}_{10} - \boldsymbol{p}_{00}$ and $\boldsymbol{p}_{01} - \boldsymbol{p}_{00}$ and where $\mathbf{0}^\mathsf{T}$ is the 1×2 vector of zeros. Equation (2) defines the 3×3 matrix A_p. The inverse of the affine transformation maps $\boldsymbol{p}$ to $\boldsymbol{x}$,

$$\left[\begin{array}{c} \boldsymbol{x} \\ \hline 1 \end{array}\right] = A_p^{-1} \left[\begin{array}{c} \boldsymbol{p} \\ \hline 1 \end{array}\right] = \left[\begin{array}{c|c} M_p^{-1} & -M_p^{-1}\boldsymbol{p}_{00} \\ \hline \mathbf{0}^\mathsf{T} & 1 \end{array}\right] \left[\begin{array}{c} \boldsymbol{p} \\ \hline 1 \end{array}\right] = \left[\begin{array}{c} M_p^{-1}(\boldsymbol{p} - \boldsymbol{p}_{00}) \\ \hline 1 \end{array}\right] \tag{3}$$

Similarly, we can write points $\boldsymbol{q}$ in the target quadrilateral as

$$\boldsymbol{q} = \boldsymbol{q}_{00} + y_0(\boldsymbol{q}_{10} - \boldsymbol{q}_{00}) + y_1(\boldsymbol{q}_{01} - \boldsymbol{q}_{00}) \tag{4}$$

The representation of $\boldsymbol{q}_{11}$ leads to $(y_0, y_1) = (c, d)$, which is shown in the lower-right quadrilateral of figure 1. Define $\boldsymbol{y} = (y_0, y_1)$. In homogeneous coordinates,

$$\left[\begin{array}{c} \boldsymbol{q} \\ \hline 1 \end{array}\right] = \left[\begin{array}{cc|c} \boldsymbol{q}_{10} - \boldsymbol{q}_{00} & \boldsymbol{q}_{01} - \boldsymbol{q}_{00} & \boldsymbol{q}_{00} \\ \hline 0 & 0 & 1 \end{array}\right] \left[\begin{array}{c} \boldsymbol{y} \\ \hline 1 \end{array}\right] = \left[\begin{array}{c|c} M_q & \boldsymbol{q}_{00} \\ \hline \mathbf{0}^\mathsf{T} & 1 \end{array}\right] \left[\begin{array}{c} \boldsymbol{y} \\ \hline 1 \end{array}\right] = A_q \left[\begin{array}{c} \boldsymbol{y} \\ \hline 1 \end{array}\right] \tag{5}$$

and

$$\left[\begin{array}{c} \boldsymbol{y} \\ \hline 1 \end{array}\right] = A_q^{-1} \left[\begin{array}{c} \boldsymbol{q} \\ \hline 1 \end{array}\right] = \left[\begin{array}{c|c} M_q^{-1} & -M_q^{-1}\boldsymbol{p}_{00} \\ \hline \mathbf{0}^\mathsf{T} & 1 \end{array}\right] \left[\begin{array}{c} \boldsymbol{q} \\ \hline 1 \end{array}\right] = \left[\begin{array}{c} M_q^{-1}(\boldsymbol{q} - \boldsymbol{q}_{00}) \\ \hline 1 \end{array}\right] \tag{6}$$

where M_q is a 2×2 matrix whose columns are $\boldsymbol{q}_{10} - \boldsymbol{q}_{00}$ and $\boldsymbol{q}_{01} - \boldsymbol{q}_{00}$. The equation defines the 3×3 matrix A_q. The inverse of the affine transformation maps $\boldsymbol{q}$ to $\boldsymbol{y}$,

In Figure 1, the lower-left and lower-right quadrilaterals are called *canonical quadrilaterals*. The first is mapped perspectively to the second using a fractional linear transformation,

$$(y_0, y_1) = f(x_0, x_1) = \frac{(f_{00}x_0 + f_{01}x_1 + f_{02}, f_{10}x_0 + f_{11}x_1 + f_{12})}{f_{20}x_0 + f_{21}x_1 + 1} \tag{7}$$

The 8 unknown coefficients f_{ij} must be determined from the 4 pairs of related 2-tuple vertices, which leads to a system of 8 linear equations in 8 unknowns.

The vertex $\boldsymbol{x} = (0,0)$ maps to $\boldsymbol{y} = (0,0)$, which leads to $f_{02} = 0$ and $f_{12} = 0$. The vertex $(1,0)$ maps to $(1,0)$, which leads to $f_{00}/(f_{20}+1) = 1$ and $f_{10}/(f_{20}+1) = 0$; therefore, $f_{00} = f_{20} + 1$ and $f_{10} = 0$. The vertex $(0,1)$ maps to $(0,1)$, which leads to $f_{01}/(f_{21}+1) = 0$ and $f_{11}/(f_{21}+1) = 1$; therefore, $f_{01} = 0$ and $f_{11} = f_{21}+1$. Finally, vertex (a,b) maps to (c,d), which leads to the linear equations $f_{00}a = c(f_{20}a + f_{21}b + 1)$ and $f_{11}b = d(f_{20}a + f_{21}b + 1)$. The fractional linear coefficients are

$$\begin{array}{llll} f_{00} = \frac{c(a+b-1)}{a(c+d-1)}, & f_{11} = \frac{d(a+b-1)}{b(c+d-1)}, & f_{20} = \frac{a-c+bc-ad}{a(c+d-1)}, & f_{21} = \frac{b-d-bc+ad}{b(c+d-1)}, \\ f_{01} = 0, & f_{02} = 0, & f_{10} = 0, & f_{12} = 0 \end{array} \tag{8}$$

Substituting these in equation (7), multiply numerator and denominator by $ab(c+d-1)$, and factor $(a-c+bc-ad) = c(a+b-1) - a(c+d-1)$ and $(b-d-bc+ad) = d(a+b-1) - b(c+d-1)$ leads to

$$(y_0, y_1) = \frac{(bc(a+b-1)x_0, ad(a+b-1)x_1)}{b(c(a+b-1) - a(c+d-1))x_0 + a(d(a+b-1) - b(c+d-1))x_1 + ab(c+d-1)} \tag{9}$$

The convexity of the $\boldsymbol{p}$-quadrilateral is characterized by $a+b-1>0$ and implies that both numerator coefficients $bc(a+b-1)$ and $ad(a+b-1)$ are positive. The denominator is a function of the form $k_0x_0+k_1x_1+k_2$ where $(x_0, x_1)$is in the canonical quadrilateral shown in figure 1. The containment constraints are $x_0 \geq 0$, $x_1 \geq 0$, $(1-b)x_0 + a(x_1-1) \leq 0$ and $b(x_0-1) + (1-a)x_1 \leq 0$. The minimum of the denominator subject to the linear inequality constraints is solved as a linear programming problem that is easily solved. The minimum must occur at a vertex of the domain. The denominator at $(0,0)$ is $ab(c+d-1)$, which is positive because the convexity of the $\boldsymbol{q}$-quadrilateral is characterized by $c+d-1>0$. The denominator at $(1,0)$ is $bc(a+b-1)$, which is positive. The denominator at $(0,1)$ is $ad(a+b-1)$, which is positive. Finally, the denominator at (a,b) is $ab(a+b-1)$, which is positive. The minimum of the denominator is positive, which means equation (9) has no singularities in its domain.

The fractional linear transformation from the $\boldsymbol{q}$-quadrilateral to the $\boldsymbol{p}$-quadrilateral is the inverse of the function in equation (9),

$$(x_0, x_1) = \frac{(da(c+d-1)y_0, cb(c+d-1)y_1)}{d(a(c+d-1) - c(a+b-1))y_0 + c(b(c+d-1) - d(a+b-1))y_1 + cd(a+b-1)} \tag{10}$$

This may be constructed by simply swapping (x_0, x_1) and (y_0, y_1) and by swapping (a, b) and (c, d) in equation (9).

The perspective mapping from the $\boldsymbol{p}$-quadrilateral to the $\boldsymbol{q}$-quadrilateral is provided by a 3×3 homogeneous matrix and a perspective divide. Define $s = a+b-1$ and $t = c+d-1$. The homogeneous matrix is

$$F = \left[\begin{array}{cc|c} bcs & 0 & 0 \\ 0 & ads & 0 \\ \hline b(cs-at) & a(ds-bt) & abt \end{array}\right] = \left[\begin{array}{c|c} D & \mathbf{0} \\ \hline \boldsymbol{\ell}^{\mathsf{T}} & \lambda \end{array}\right] \tag{11}$$

where D is a 2×2 diagonal matrix, $\mathbf{0}$ is the 2×1 vector of zeros, $\boldsymbol{\ell}^{\mathsf{T}}$ is a 1×2 vector and λ is a scalar. The perspective mapping is

$$\left[\begin{array}{c} \boldsymbol{q} \\ \hline 1 \end{array}\right] \sim \left[\begin{array}{c} \boldsymbol{q}' \\ \hline w \end{array}\right] = A_q F A_p^{-1} \left[\begin{array}{c} \boldsymbol{q} \\ \hline 1 \end{array}\right] \tag{12}$$

where the similarity symbol indicates that the left-hand side is obtained via the perspective divide: $\boldsymbol{q} = \boldsymbol{q}'/w$. The 3×3 homography matrix $H = A_q F A_p^{-1}$ is

$$\begin{aligned} H &= \left[\begin{array}{c|c} M_q & \boldsymbol{q}_{00} \\ \hline \mathbf{0}^{\mathsf{T}} & 1 \end{array}\right] \left[\begin{array}{c|c} D & \mathbf{0} \\ \hline \boldsymbol{\ell}^{\mathsf{T}} & \lambda \end{array}\right] \left[\begin{array}{c|c} M_p^{-1} & -M_p^{-1}\boldsymbol{p}_{00} \\ \hline \mathbf{0}^{\mathsf{T}} & 1 \end{array}\right] \\ &= \left[\begin{array}{c|c} (M_q D + \boldsymbol{q}_{00}\boldsymbol{\ell}^{\mathsf{T}})M_p^{-1} & -(M_q D + \boldsymbol{q}_{00}\boldsymbol{\ell}^{\mathsf{T}})M_p^{-1}\boldsymbol{p}_{00} + \lambda\boldsymbol{q}_{00} \\ \hline \boldsymbol{\ell}^{\mathsf{T}} M_p^{-1} & -\boldsymbol{\ell}^{\mathsf{T}} M_p^{-1}\boldsymbol{p}_{00} + \lambda \end{array}\right] \end{aligned} \tag{13}$$

Some algebra will show that

$$H\begin{bmatrix}\boldsymbol{p}_{00}\\ \hline 1\end{bmatrix}=\begin{bmatrix}abt\,\boldsymbol{q}_{00}\\ \hline abt\end{bmatrix},\ H\begin{bmatrix}\boldsymbol{p}_{10}\\ \hline 1\end{bmatrix}=\begin{bmatrix}bcs\,\boldsymbol{q}_{10}\\ \hline bcs\end{bmatrix},\ H\begin{bmatrix}\boldsymbol{p}_{01}\\ \hline 1\end{bmatrix}=\begin{bmatrix}ads\,\boldsymbol{q}_{01}\\ \hline ads\end{bmatrix},\ H\begin{bmatrix}\boldsymbol{p}_{11}\\ \hline 1\end{bmatrix}=\begin{bmatrix}abs\,\boldsymbol{q}_{00}\\ \hline abs\end{bmatrix} \tag{14}$$

The perspective division leads to the correct mapping of vertices of the $\boldsymbol{p}$-quadrilateral to the $\boldsymbol{q}$-quadrilateral. Generally, the mapping of $\boldsymbol{p}$ to $\boldsymbol{q}$ is

$$\boldsymbol{q}=\frac{(M_qD+\boldsymbol{q}_{00}\boldsymbol{\ell}^{\mathsf{T}})M_p^{-1}(\boldsymbol{p}-\boldsymbol{p}_{00})+\lambda\boldsymbol{q}_{00}}{\boldsymbol{\ell}^{\mathsf{T}}M_p^{-1}(\boldsymbol{p}-\boldsymbol{p}_{00})+\lambda} \tag{15}$$

Consider the special case when the quadrilaterals are canonical. First, let the source quadrilateral be the unit square so that $(a,b)=(1,1)$. Equation (9) reduces to

$$(y_0,y_1)=\frac{(cx_0,dx_1)}{(1-d)x_0+(1-c)x_1+(c+d-1)} \tag{16}$$

Second, let thee target quadrilateral be the unit square so that $(c,d)=(1,1)$. Equation (10) reduces to

$$(x_0,x_1)=\frac{(ay_0,by_1)}{(1-b)y_0+(1-a)y_1+(a+b-1)} \tag{17}$$

3 Mapping Between Cuboids

In 3D, a cuboid is a convex polyhedron with 6 convex quadrilateral faces. Let the vertices be $\boldsymbol{p}_i$ for $0\le i\le 7$. The cuboid faces are specified by 4-tuples of indices, each face having vertices in counterclockwise order when viewed by an observer outside the cuboid. The faces are $F_0=\{0,2,3,1\}$, $F_1=\{0,4,6,2\}$, $F_2=\{0,1,5,4\}$, $F_3=\{7,5,1,3\}$, $F_4=\{7,3,2,6\}$, and $F_5=\{7,6,4,5\}$. For example, the unit cube is a special cuboid with $\boldsymbol{p}_0=(0,0,0)$, $\boldsymbol{p}_1=(1,0,0)$, $\boldsymbol{p}_2=(0,1,0)$, $\boldsymbol{p}_3=(1,1,0)$, $\boldsymbol{p}_4=(0,0,1)$, $\boldsymbol{p}_5=(1,0,1)$, $\boldsymbol{p}_6=(0,1,1)$, and $\boldsymbol{p}_7=(1,1,1)$.

There appears to be no perspective mapping that applies to all pairs of cuboids. This section describes the attempt to construct such a mapping in order to understand why it fails for all pairs.[1]

3.1 Reduction to Canonical Cuboids

As in the perspective mapping between a pair of convex quadrilaterals, it is sufficient to use affine transformations to convert the cuboids to a canonical form. Define a *canonical cuboid* to have vertices $\boldsymbol{u}_0=(0,0,0)$, $\boldsymbol{u}_1=(1,0,0)$, $\boldsymbol{u}_2=(0,1,0)$, $\boldsymbol{u}_3=(a_3,b_3,0)$, $\boldsymbol{u}_4=(0,0,1)$, $\boldsymbol{u}_5=(a_5,0,c_5)$, $\boldsymbol{u}_6=(0,b_6,c_6)$, and $\boldsymbol{u}_7=(a_7,b_7,c_7)$. A general cuboid can be affinely transformed to a canonical cuboid by translating $\boldsymbol{p}_0$ to the origin. The vectors $\boldsymbol{p}_1-\boldsymbol{p}_0$, $\boldsymbol{p}_2-\boldsymbol{p}_0$, and $\boldsymbol{p}_4$ are transformed to $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$, respectively.

As a consequence of the translation and mapping of source edges to the coordinate axes, the vector $\boldsymbol{p}_3-\boldsymbol{p}_0$ is transformed into the plane $z=0$ as a point $(a_3,b_3,0)$. The condition $z=0$ is based on the general cuboid

[1]Thanks to Dennis Gustafsson of Tuxedo Labs for reporting that the algorithm of the previous version (April 1, 2019) of the PDF failed for his data and stating that the mapping appeared not to depend on $\boldsymbol{p}_3$, $\boldsymbol{p}_5$ or $\boldsymbol{p}_6$ when it should.

having a planar face $\{0, 2, 3, 1\}$. The face must be a convex quadrilateral, which forces $a_3 + b_3 \geq 1$. Similarly, $\boldsymbol{p}_5 - \boldsymbol{p}_0$ is transformed into the plane $y = 0$ as a point $(a_5, 0, c_5)$ with $a_5 + c_5 \geq 1$ and $\boldsymbol{p}_6$ is transformed into the plane $x = 0$ as a point $(0, b_6, c_6)$ with $b_6 + c_6 \geq 1$.

The vector $\boldsymbol{p}_7 - \boldsymbol{p}_0$ is transformed to (a_7, b_7, c_7). The convexity of the general cuboid implies the convexity of the canonical cuboid. The vertices $\boldsymbol{u}_0$, $\boldsymbol{u}_1$, $\boldsymbol{u}_2$, and $\boldsymbol{u}_3$ are those of a tetrahedron. For the canonical cuboid to be convex and have 6 planar faces for which no 2 adjacent faces are coplanar, it is necessary that $a_7 + b_7 + c_7 > 1$.

Faces $F_0 = \{0, 2, 3, 1\}$, $F_1 = \{0, 4, 6, 2\}$, and $F_2 = \{0, 1, 5, 4\}$ are planar. For face $F_3 = \{7, 5, 1, 3\}$ to be planar, $\boldsymbol{v}_7 - \boldsymbol{v}_1$ must be in the span of $\boldsymbol{v}_3 - \boldsymbol{v}_1$ and $\boldsymbol{v}_5 - \boldsymbol{v}_1$; therefore,

$$0 = (\boldsymbol{v}_7 - \boldsymbol{v}_1) \cdot (\boldsymbol{v}_3 - \boldsymbol{v}_1) \times (\boldsymbol{v}_5 - \boldsymbol{v}_1) = (-1 + a_7) b_3 c_5 + (1 - a_3) b_7 c_5 + b_3 c_7 (1 - a_5) \tag{18}$$

For face $F_4 = \{7, 3, 2, 6\}$ to be planar, $\boldsymbol{v}_7 - \boldsymbol{v}_2$ must be in the span of $\boldsymbol{v}_6 - \boldsymbol{v}_2$ and $\boldsymbol{v}_3 - \boldsymbol{v}_2$; therefore,

$$0 = (\boldsymbol{v}_7 - \boldsymbol{v}_2) \cdot (\boldsymbol{v}_6 - \boldsymbol{v}_2) \times (\boldsymbol{v}_3 - \boldsymbol{v}_2) = a_3 (-1 + b_7) c_6 + a_7 (1 - b_3) c_6 + a_3 (1 - b_6) c_7 \tag{19}$$

For face $F_5 = \{7, 6, 4, 5\}$ to be planar, $\boldsymbol{v}_7 - \boldsymbol{v}_4$ must be in the span of $\boldsymbol{v}_5 - \boldsymbol{v}_4$ and $\boldsymbol{v}_6 - \boldsymbol{v}_4$; therefore,

$$0 = (\boldsymbol{v}_7 - \boldsymbol{v}_4) \cdot (\boldsymbol{v}_5 - \boldsymbol{v}_4) \times (\boldsymbol{v}_6 - \boldsymbol{v}_4) = a_7 b_6 (1 - c_5) + a_5 b_7 (1 - c_6) + a_5 b_6 (-1 + c_7) \tag{20}$$

If a_3, b_3, a_5, c_5, b_6, and c_6 are specified, then (a_7, b_7, c_7) is the solution to a linear system defined by equations (18), (19), and (20),

$$\begin{bmatrix} b_3 c_5 & (1 - a_3) c_5 & (1 - a_5) b_3 \\ (1 - b_3) c_6 & a_3 c_6 & a_3 (1 - b_6) \\ b_6 (1 - c_5) & a_5 (1 - c_6) & a_5 b_6 \end{bmatrix} \begin{bmatrix} a_7 \\ b_7 \\ c_7 \end{bmatrix} = \begin{bmatrix} b_3 c_5 \\ a_3 c_6 \\ a_5 b_6 \end{bmatrix} \tag{21}$$

3.2 Fractional Linear Transformation for Canonical Cuboids

The fractional linear transformation for cuboids must be of the form

$$\begin{aligned} (y_0, y_1, y_2) &= f(x_0, x_1, x_2) \\ &= \frac{(f_{00} x_0 + f_{01} x_1 + f_{02} x_2 + f_{03}, f_{10} x_0 + f_{11} x_1 + f_{12} x_2 + f_{13}, f_{20} x_0 + f_{21} x_1 + f_{22} x_2 + f_{23})}{f_{30} x_0 + f_{31} x_1 + f_{32} x_2 + 1} \end{aligned} \tag{22}$$

Generally, mapping 8 source 3-tuple vertices to 8 target 3-tuple vertices leads to a linear system of 24 equations in 15 unknowns. One would expect that the linear system is usually overconstrained; that is, it has no solution. As we will see, sometimes it is not overconstrained and has a solutions.

To simplify the presentation, let the source canonical cuboid be the unit cube with vertices $\boldsymbol{u}_0 = (0, 0, 0)$, $\boldsymbol{u}_1 = (1, 0, 0)$, $\boldsymbol{u}_2 = (0, 1, 0)$, $\boldsymbol{u}_3 = (1, 1, 0)$, $\boldsymbol{u}_4 = (0, 0, 1)$, $\boldsymbol{u}_5 = (1, 0, 1)$, $\boldsymbol{u}_6 = (0, 1, 1)$, and $\boldsymbol{u}_7 = (1, 1, 1)$. Let the target canonical cuboid have vertices $\boldsymbol{v}_0 = (0, 0, 0)$, $\boldsymbol{v}_1 = (1, 0, 0)$, $\boldsymbol{v}_2 = (0, 1, 0)$, $\boldsymbol{v}_3 = (a_3, b_3, 0)$,

$\boldsymbol{v}_4 = (0,0,1)$, $\boldsymbol{v}_5 = (a_5, 0, c_5)$, $\boldsymbol{v}_6 = (0, b_6, c_6)$, and $\boldsymbol{v}_7 = (a_7, b_7, c_7)$. Substituting these into equation (22),

$$\begin{array}{lcl} (0,0,0) & = & (f_{03}, f_{13}, f_{23}) \\ (1,0,0) & = & (f_{00}, f_{10}, f_{20})/(f_{30}+1) \\ (0,1,0) & = & (f_{01}, f_{11}, f_{21})/(f_{31}+1) \\ (a_3, b_3, 0) & = & (f_{00}+f_{01}, f_{10}+f_{11}, f_{20}+f_{21})/(f_{30}+f_{31}+1) \\ (0,0,1) & = & (f_{02}, f_{12}, f_{22})/(f_{32}+1) \\ (a_5, 0, c_5) & = & (f_{00}+f_{02}, f_{10}+f_{12}, f_{20}+f_{22})/(f_{30}+f_{32}+1) \\ (0, b_6, c_6) & = & (f_{01}+f_{02}, f_{11}+f_{12}, f_{21}+f_{22})/(f_{31}+f_{32}+1) \\ (a_7, b_7, c_7) & = & (f_{00}+f_{01}+f_{02}, f_{10}+f_{11}+f_{12}, f_{20}+f_{21}+f_{22})/(f_{30}+f_{31}+f_{32}+1) \end{array} \tag{23}$$

A partial reduction of the equations is

$$\begin{array}{l} f_{03} = f_{13} = f_{23} = f_{10} = f_{20} = f_{01} = f_{21} = f_{02} = f_{12} = 0 \\ f_{00} = f_{30}+1, \quad f_{11} = f_{31}+1, \quad f_{22} = f_{32}+1 \\ (a_3, b_3) = (f_{00}, f_{11})/(f_{30}+f_{31}+1) = (f_{30}+1, f_{31}+1)/(f_{30}+f_{31}+1) \\ (a_5, c_5) = (f_{00}, f_{22})/(f_{30}+f_{32}+1) = (f_{30}+1, f_{32}+1)/(f_{30}+f_{32}+1) \\ (b_6, c_6) = (f_{11}, f_{22})/(f_{31}+f_{32}+1) = (f_{31}+1, f_{32}+1)/(f_{31}+f_{32}+1) \\ (a_7, b_7, c_7) = (f_{00}, f_{11}, f_{22})/(f_{30}+f_{31}+f_{32}+1) = (f_{30}+1, f_{31}+1, f_{32}+1)/(f_{30}+f_{31}+f_{32}+1) \end{array} \tag{24}$$

The last 4 of these represent 9 linear equations involving known quantities $(a_3, b_3, a_5, c_5, b_6, c_6, a_7, b_7, c_7)$ with 3 unknowns f_{30}, f_{31}, and f_{32}. For independent known quantities, the linear system is generally overconstrained. However, the planar-faces constraints of equations (18), (19), and (20) force dependencies among the a-, b-, and c-components.

The planar-face constraints can be formulated in terms of the f_{ij} rather than the a-, b-, and c-components,

$$\begin{array}{lclcl} (\boldsymbol{v}_7 - \boldsymbol{v}_1) \cdot (\boldsymbol{v}_3 - \boldsymbol{v}_1) \times (\boldsymbol{v}_5 - \boldsymbol{v}_1) & = & (f_{12} - f_{21}) n_0 / d & = & 0 \\ (\boldsymbol{v}_7 - \boldsymbol{v}_2) \cdot (\boldsymbol{v}_6 - \boldsymbol{v}_2) \times (\boldsymbol{v}_3 - \boldsymbol{v}_2) & = & (f_{12} - f_{21}) n_1 / d & = & 0 \\ (\boldsymbol{v}_7 - \boldsymbol{v}_4) \cdot (\boldsymbol{v}_5 - \boldsymbol{v}_4) \times (\boldsymbol{v}_6 - \boldsymbol{v}_4) & = & (f_{12} - f_{21}) n_2 / d & = & 0 \end{array} \tag{25}$$

where the values of n_0, n_1, n_2, and d depend on the f_{ij} but are irrelevant because $f_{12} = 0$ and $f_{21} = 0$. If there is a solution for the coefficients f_{ij} of the fractional linear transformation, then necessarily the planar-face constraints are satisfied.

3.3 Example Where There is a Perspective Mapping

The equation

$$(a_7, b_7, c_7) = (f_{30}+1, f_{31}+1, f_{32}+1)/(f_{30}+f_{31}+f_{32}+1) \tag{26}$$

can be solved for f_{30}, f_{31}, and f_{32},

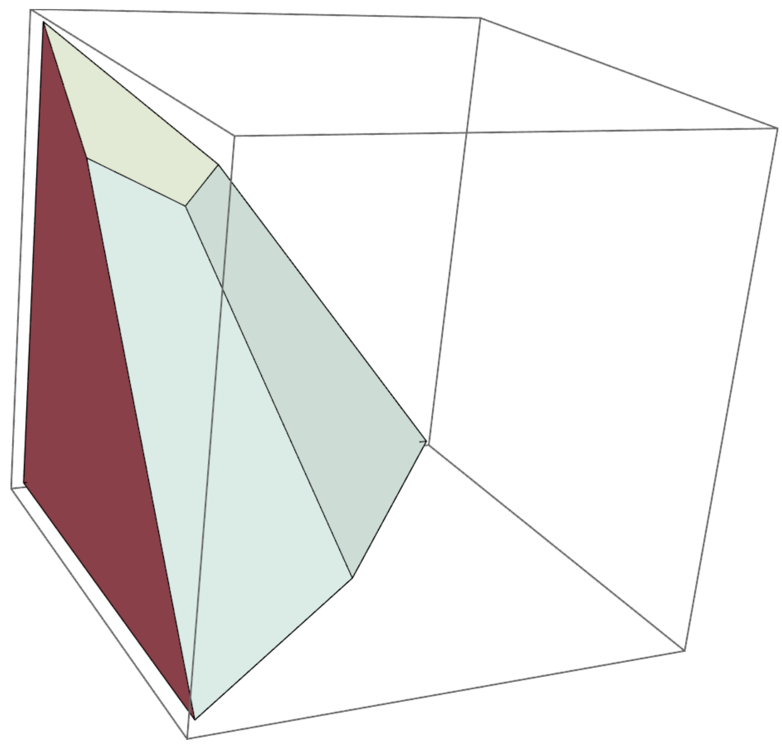
$$(f_{00}, f_{11}, f_{22}, f_{30}, f_{31}, f_{32}) = \frac{(2a_7, 2b_7, 2c_7, 1 + a_7 - b_7 - c_7, 1 - a_7 + b_7 - c_7, 1 - a_7 - b_7 + c_7)}{-1 + a_7 + b_7 + c_7} \tag{27}$$

However, the solution forces

$$(a_3, b_3) = \frac{(2a_7, 2b_7)}{1 + a_7 + b_7 - c_7}, \quad (a_5, c_5) = \frac{(2a_7, 2c_7)}{1 + a_7 - b_7 + c_7}, \quad (b_6, c_6) = \frac{(2b_7, 2c_7)}{1 - a_7 + b_7 + c_7} \tag{28}$$

Choose $(a_7, b_7, c_7) = (18, 20, 56)/83$; then $(a_3, b_3) = (36, 40)/65$, $(a_5, c_5) = (36, 112)/137$, and $(b_6, c_6) = (40, 112)/141$. Figure 2 shows a rendering of the cuboid.

Figure 2. The cuboid of this section, rendered with Mathematica [1].

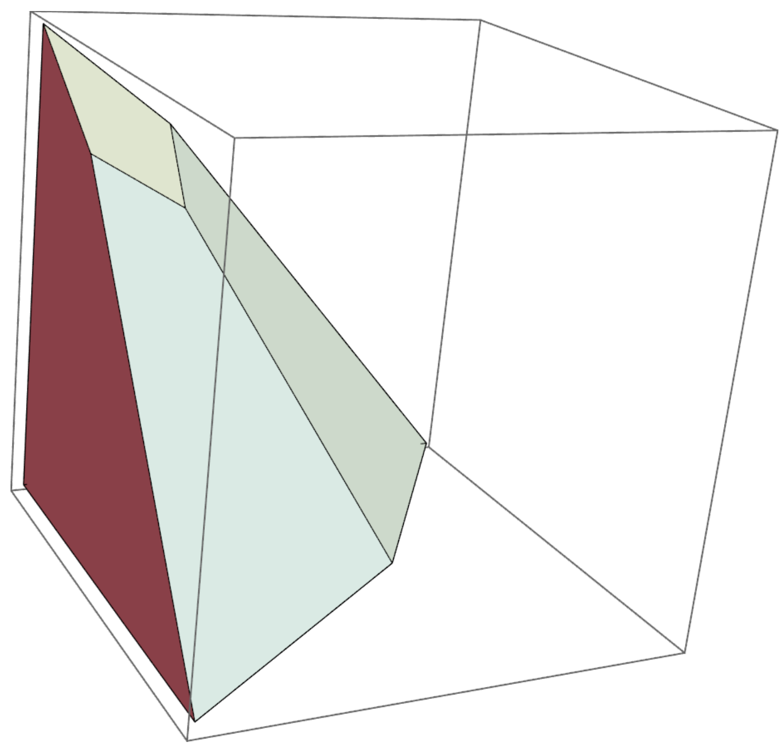


3.4 Example Where There is No Perspective Mapping

Let $(a_3, b_3, 0) = (6, 5, 0)/10$, $(a_5, 0, c_5) = (3, 0, 8)/10$, and $(0, b_6, c_6) = (0, 4, 7)/10$. The planar-face constraints have solution $(a_7, b_7, c_7) = (18, 20, 56)/83$, which is the same-named vertex of the previous section where there does exist a perspective mapping. The faces are planar.

Figure 3 shows a rendering of the cuboid.

Figure 3. The cuboid of this section, rendered with Mathematica [1].



However, the system of equations for f_{ij} has no solution, which was verified using Mathematics [1].

4 Perspective Mapping of a View Frustum to a Cube

In computer graphics, world space is mapped to camera space. The latter space has the camera location $\boldsymbol{E}$ (eyepoint) and coordinate axes $\boldsymbol{D}$ (view direction), $\boldsymbol{U}$ (up direction), and $\boldsymbol{R}$ (right direction). The set $\{\boldsymbol{D}, \boldsymbol{U}, \boldsymbol{R}\}$ is a right-handed orthonormal basis; that is, the axis directions are unit length, mutually perpendicular, and $\boldsymbol{D} = \boldsymbol{U} \times \boldsymbol{R}$. However, the standard treatment of camera space is that it has a left-handed coordinate system: $\{\boldsymbol{R}, \boldsymbol{U}, \boldsymbol{D}\}$. A world point $\boldsymbol{P}$ is represented by

$$\boldsymbol{P} = \boldsymbol{E} + r\boldsymbol{R} + u\boldsymbol{U} + d\boldsymbol{D} \tag{29}$$

for $r = \boldsymbol{R} \cdot (\boldsymbol{P} - \boldsymbol{E})$, $u = \boldsymbol{U} \cdot (\boldsymbol{P} - \boldsymbol{E})$, and $d = \boldsymbol{U} \cdot (\boldsymbol{P} - \boldsymbol{E})$. The 3-tuple (r, u, d) is referred to as the camera-space coordinates of $\boldsymbol{P}$.

A view frustum defines a region of world space. A scene is culled and clipped so that only that portion inside the symmetric view frustum is rendered. The view frustum is a cuboid with 6 planar faces labeled as *near*, *far*, *right*, *left*, *top*, and *bottom*. The near face and far face are parallel. The near face is $d_{\min} > 0$ units from the camera eyepoint measured in the view direction. The far face is $d_{\max} > d_{\min}$ units from the camera

eyepoint measured in the view direction. The other faces are defined by the constraints $r \in [r_{\min}, +r_{\max}]$ and $u \in [u_{\min}, u_{\max}]$ for extreme values satisfying $r_{\max} > r_{\min}$ and $u_{\max} > u_{\min}$. The view frustum is said to be symmetric when $r_{\min} = -r_{\max}$ and $u_{\min} = -u_{\max}$.

The view frustum points $(x_0, x_1, x_2) = (r, u, d)$ are mapped to a cube with points (y_0, y_1, y_2). For DirectX graphics, the cube is $[-1, 1] \times [-1, 1] \times [0, 1]$. For OpenGL graphics, the cube is $[-1, 1] \times [-1, 1] \times [-1, 1]$. The near-face vertices are $(r_{\min}, u_{\min}, d_{\min})$, $(r_{\max}, u_{\min}, d_{\min})$, $(r_{\min}, u_{\max}, d_{\min})$, and $(r_{\max}, u_{\max}, d_{\min})$. The far-face vertices are obtain from the near-face vertices by multiplying by $d_{\max}/d_{\min}$. The correspondences for view frustum vertices and cube vertices are shown in table 1

Table 1. The mapping of view frustum vertices to cube vertices.

frustum vertex	DirectX cube	OpenGL cube
$(r_{\min}, u_{\min}, d_{\min})$	$(-1, -1, 0)$	$(-1, -1, -1)$
$(r_{\max}, u_{\min}, d_{\min})$	$(+1, -1, 0)$	$(+1, -1, -1)$
$(r_{\min}, u_{\max}, d_{\min})$	$(-1, +1, 0)$	$(-1, +1, -1)$
$(r_{\max}, u_{\max}, d_{\min})$	$(+1, +1, 0)$	$(+1, +1, -1)$
$(d_{\max}/d_{min})(r_{\min}, u_{\min}, d_{\min})$	$(-1, -1, +1)$	$(-1, -1, +1)$
$(d_{\max}/d_{min})(r_{\max}, u_{\min}, d_{\min})$	$(+1, -1, +1)$	$(+1, -1, +1)$
$(d_{\max}/d_{min})(r_{\min}, u_{\max}, d_{\min})$	$(-1, +1, +1)$	$(-1, +1, +1)$
$(d_{\max}/d_{min})(r_{\max}, u_{\max}, d_{\min})$	$(+1, +1, +1)$	$(+1, +1, +1)$

Documentation for the graphics APIs typically define the homogeneous matrices for the perspective mapping. Because the graphics APIs use left-handed camera coordinates, the second column is negated so that it corresponds to $-\boldsymbol{D}$ rather than $\boldsymbol{D}$. For DirectX it is

$$P_{[0,1]} = \left[\begin{array}{ccr|r} \frac{2d_{\min}}{r_{\max}-r_{\min}} & 0 & -\frac{r_{\max}+r_{\min}}{r_{\max}-r_{\min}} & 0 \\ 0 & \frac{2d_{\min}}{u_{\max}-u_{\min}} & -\frac{u_{\max}+u_{\min}}{u_{\max}-u_{\min}} & 0 \\ 0 & 0 & \frac{d_{\max}}{d_{\max}-d_{\min}} & -\frac{d_{\min}d_{\max}}{d_{\max}-d_{\min}} \\ \hline 0 & 0 & 1 & 0 \end{array}\right] \tag{30}$$

For OpenGL it is

$$P_{[-1,1]} = \left[\begin{array}{ccr|r} \frac{2d_{\min}}{r_{\max}-r_{\min}} & 0 & -\frac{r_{\max}+r_{\min}}{r_{\max}-r_{\min}} & 0 \\ 0 & \frac{2d_{\min}}{u_{\max}-u_{\min}} & -\frac{u_{\max}+u_{\min}}{u_{\max}-u_{\min}} & 0 \\ 0 & 0 & \frac{d_{\max}+d_{\min}}{d_{\max}-d_{\min}} & -\frac{2d_{\min}d_{\max}}{d_{\max}-d_{\min}} \\ \hline 0 & 0 & 1 & 0 \end{array}\right] \tag{31}$$

The mapping with the perspective divide is a tuple of fractional linear transformations similar to the equation (22). However, the mapping from view frustum to cube is constructed here directly without an intermediate

canonical cuboid. Let $\boldsymbol{f}_i = (f_{i0}, f_{i1}, f_{i2}, f_{i3})$ for $0 \leq i \leq 3$. Define $\boldsymbol{x} = (x_0, x_1, x_2)$ and $\boldsymbol{y} = (y_0, y_1, y_2)$. The transformations are

$$(\boldsymbol{f}_0 \cdot (\boldsymbol{x}, 1), \boldsymbol{f}_1 \cdot (\boldsymbol{x}, 1), \boldsymbol{f}_2 \cdot (\boldsymbol{x}, 1)) - (\boldsymbol{f}_3 \cdot (\boldsymbol{x}, 1))\, \boldsymbol{y} = \boldsymbol{0} \tag{32}$$

The 8 vertices of the view frustum are mapped to the corresponding 8 vertices of the cube. Equation (32) represents a linear system of equations with 16 variables f_{ij} and 24 equations. As indicated previously, there might not be a solution when mapping general cuboids to cuboids. However, for the mapping of a view frustum to a cube, the linear system has a 1-parameter family of solutions. Any nonzero choice of the arbitrary parameter is acceptable because the parameter is removed by the perspective divide.

I used Mathematica [1] to solve the linear system in equation (32). The code is shown in listing 1.

Listing 1. Mathematics [1] code for constructing the perspective mappings. For DirectX, `r0` is $r_{\min}$, `r1` is $r_{\max}$, `u0` is $u_{\min}$, `u1` is $u_{\max}$, `d0` is $d_{\min}$, and `d1` is $d_{\max}$.

```
p0 = {f00, f01, f02, f03}
p1 = {f10, f11, f12, f13}
p2 = {f20, f21, f22, f23}
p3 = {f30, f31, f32, f33}
eqn[x0_, x1_, x2_, y0_, y1_, y2_] =
    {Dot[p0, {x0, x1, x2, 1}], Dot[p1, {x0, x1, x2, 1}], Dot[p2, {x0, x1, x2, 1}]} − {y0, y1, y2}∗Dot[p3, {x0, x1, x2, 1}]

(∗ View frustum to DirectX cube [−1,1]x[−1,1]x[0,1] ∗)
e1 = eqn[r0, u0, d0, −1, −1, 0]
e2 = eqn[r1, u0, d0, 1, −1, 0]
e3 = eqn[r0, u1, d0, −1, 1, 0]
e4 = eqn[r1, u1, d0, 1, 1, 0]
e5 = eqn[r0∗d1/d0, u0∗d1/d0, d1, −1, −1, 1]
e6 = eqn[r1∗d1/d0, u0∗d1/d0, d1, 1, −1, 1]
e7 = eqn[r0∗d1/d0, u1∗d1/d0, d1, −1, 1, 1]
e8 = eqn[r1∗d1/d0, u1∗d1/d0, d1, 1, 1, 1]
Simplify[Solve[
    e1 == {0, 0, 0} && e2 == {0, 0, 0} && e3 == {0, 0, 0} &&
    e4 == {0, 0, 0} && e5 == {0, 0, 0} && e6 == {0, 0, 0} &&
    e7 == {0, 0, 0} && e8 == {0, 0, 0} && f22 == d1/(d1 − d0),
    {f00, f01, f02, f03, f10, f11, f12, f13, f20, f21, f22, f23, f30, f31, f32, f33}]]

{{
  f00 −> −((2 d0)/(r0 − r1)), f01 −> 0, f02 −> (r0 + r1)/(r0 − r1), f03 −> 0,
  f10 −> 0, f11 −> −((2 d0)/(u0 − u1)), f12 −> (u0 + u1)/(u0 − u1), f13 −> 0,
  f20 −> 0, f21 −> 0, f22 −> d1/(−d0 + d1), f23 −> (d0 d1)/(d0 − d1),
  f30 −> 0, f31 −> 0, f32 −> 1, f33 −> 0
}}

(∗ View frustum to OpenGL cube [−1,1]x[−1,1]x[−1,1] ∗)
e1 = eqn[r0, u0, d0, −1, −1, −1]
e2 = eqn[r1, u0, d0, 1, −1, −1]
e3 = eqn[r0, u1, d0, −1, 1, −1]
e4 = eqn[r1, u1, d0, 1, 1, −1]
e5 = eqn[r0∗d1/d0, u0∗d1/d0, d1, −1, −1, 1]
e6 = eqn[r1∗d1/d0, u0∗d1/d0, d1, 1, −1, 1]
e7 = eqn[r0∗d1/d0, u1∗d1/d0, d1, −1, 1, 1]
e8 = eqn[r1∗d1/d0, u1∗d1/d0, d1, 1, 1, 1]
Simplify[Solve[
    e1 == {0, 0, 0} && e2 == {0, 0, 0} && e3 == {0, 0, 0} &&
    e4 == {0, 0, 0} && e5 == {0, 0, 0} && e6 == {0, 0, 0} &&
    e7 == {0, 0, 0} && e8 == {0, 0, 0} && f22 == (d1 + d0)/(d1 − d0),
    {f00, f01, f02, f03, f10, f11, f12, f13, f20, f21, f22, f23, f30, f31, f32, f33}]]

{{
  f00 −> −((2 d0)/(r0 − r1)), f01 −> 0, f02 −> (r0 + r1)/(r0 − r1), f03 −> 0,
  f10 −> 0, f11 −> −((2 d0)/(u0 − u1)), f12 −> (u0 + u1)/(u0 − u1), f13 −> 0,
  f20 −> 0, f21 −> 0, f22 −> −((d0 + d1)/(d0 − d1)), f23 −> (2 d0 d1)/(d0 − d1),
  f30 −> 0, f31 −> 0, f32 −> 1, f33 −> 0
}}
```

---

5 No Perspective Mapping of a View Frustum with Convex Quadrilateral Viewport

The attempt to construct a perspective mapping of a view frustum whose viewport is a non-parallelogram convex quadrilateral fails. The attempt is shown next where the target cube is $[-1,1]\times[-1,1]\times[0,1]$.

The convex-quadrilateral view frustum is a cuboid with a near face and a far face that are parallel. Let the near face have camera-space vertices $\boldsymbol{q}_{ij0} = \{(r_{ij}, u_{ij}, d_{\min})\}$ for $0 \le i \le 1$ and $0 \le j \le 1$. The vertices are in counterclockwise order when viewed from the camera eyepoint $\{\boldsymbol{q}_{00}, \boldsymbol{q}_{10}, \boldsymbol{q}_{11}, \boldsymbol{q}_{01}\}$. The far face has vertices $\boldsymbol{q}_{ij1} = (d_{\max}/d_{\min})\boldsymbol{q}_{ij0}$. The vertex correspondences are the first two columns of table 1.

I tried to use Mathematica [1] to solve for the coefficients of the fractional linear transformations, but some known mathematical conditions and the presence of more equations than unknowns let to a conclusion that all coefficients are 0. I manually solved the equations in steps to understand the structure of the linear system. The 24 equations in 16 unknowns is listed next. The labels are used throughout the steps.

label	equation
(1a)	$r_{00}f_{00} + u_{00}f_{01} + nf_{02} + f_{03} + (r_{00}f_{30} + u_{00}f_{31} + nf_{32} + f_{33}) = 0$
(1b)	$r_{00}f_{10} + u_{00}f_{11} + nf_{12} + f_{13} + (r_{00}f_{30} + u_{00}f_{31} + nf_{32} + f_{33}) = 0$
(1c)	$r_{00}f_{20} + u_{00}f_{21} + nf_{22} + f_{23} = 0$
(2a)	$r_{10}f_{00} + u_{10}f_{01} + nf_{02} + f_{03} - (r_{10}f_{30} + u_{10}f_{31} + nf_{32} + f_{33}) = 0$
(2b)	$r_{10}f_{10} + u_{10}f_{11} + nf_{12} + f_{13} + (r_{10}f_{30} + u_{10}f_{31} + nf_{32} + f_{33}) = 0$
(2c)	$r_{10}f_{20} + u_{10}f_{21} + nf_{22} + f_{23} = 0$
(3a)	$r_{01}f_{00} + u_{01}f_{01} + nf_{02} + f_{03} + (r_{01}f_{30} + u_{01}f_{31} + nf_{32} + f_{33}) = 0$
(3b)	$r_{01}f_{10} + u_{01}f_{11} + nf_{12} + f_{13} - (r_{01}f_{30} + u_{01}f_{31} + nf_{32} + f_{33}) = 0$
(3c)	$r_{01}f_{20} + u_{01}f_{21} + nf_{22} + f_{23} = 0$
(4a)	$r_{11}f_{00} + u_{11}f_{01} + nf_{02} + f_{03} - (r_{11}f_{30} + u_{11}f_{31} + nf_{32} + f_{33}) = 0$
(4b)	$r_{11}f_{10} + u_{11}f_{11} + nf_{12} + f_{13} - (r_{11}f_{30} + u_{11}f_{31} + nf_{32} + f_{33}) = 0$
(4c)	$r_{11}f_{20} + u_{11}f_{21} + nf_{22} + f_{23} = 0$
(5a)	$(r_{00}d/n)f_{00} + (u_{00}d/n)f_{01} + df_{02} + f_{03} + ((r_{00}d/n)f_{30} + (u_{00}d/n)f_{31} + df_{32} + f_{33}) = 0$
(5b)	$(r_{00}d/n)f_{10} + (u_{00}d/n)f_{11} + df_{12} + f_{13} + ((r_{00}d/n)f_{30} + (u_{00}d/n)f_{31} + df_{32} + f_{33}) = 0$
(5c)	$(r_{00}d/n)f_{20} + (u_{00}d/n)f_{21} + df_{22} + f_{23} - ((r_{00}d/n)f_{30} + (u_{00}d/n)f_{31} + df_{32} + f_{33}) = 0$
(6a)	$(r_{10}d/n)f_{00} + (u_{10}d/n)f_{01} + df_{02} + f_{03} - ((r_{10}d/n)f_{30} + (u_{10}d/n)f_{31} + df_{32} + f_{33}) = 0$
(6b)	$(r_{10}d/n)f_{10} + (u_{10}d/n)f_{11} + df_{12} + f_{13} + ((r_{10}d/n)f_{30} + (u_{10}d/n)f_{31} + df_{32} + f_{33}) = 0$
(6c)	$(r_{10}d/n)f_{20} + (u_{10}d/n)f_{21} + df_{22} + f_{23} - ((r_{10}d/n)f_{30} + (u_{10}d/n)f_{31} + df_{32} + f_{33}) = 0$
(7a)	$(r_{01}d/n)f_{00} + (u_{01}d/n)f_{01} + df_{02} + f_{03} + ((r_{01}d/n)f_{30} + (u_{01}d/n)f_{31} + df_{32} + f_{33}) = 0$
(7b)	$(r_{01}d/n)f_{10} + (u_{01}d/n)f_{11} + df_{12} + f_{13} - ((r_{01}d/n)f_{30} + (u_{01}d/n)f_{31} + df_{32} + f_{33}) = 0$
(7c)	$(r_{01}d/n)f_{20} + (u_{01}d/n)f_{21} + df_{22} + f_{23} - ((r_{01}d/n)f_{30} + (u_{01}d/n)f_{31} + df_{32} + f_{33}) = 0$
(8a)	$(r_{11}d/n)f_{00} + (u_{11}d/n)f_{01} + df_{02} + f_{03} - ((r_{11}d/n)f_{30} + (u_{11}d/n)f_{31} + df_{32} + f_{33}) = 0$
(8b)	$(r_{11}d/n)f_{10} + (u_{11}d/n)f_{11} + df_{12} + f_{13} - ((r_{11}d/n)f_{30} + (u_{11}d/n)f_{31} + df_{32} + f_{33}) = 0$
(8c)	$(r_{11}d/n)f_{20} + (u_{11}d/n)f_{21} + df_{22} + f_{23} - ((r_{11}d/n)f_{30} + (u_{11}d/n)f_{31} + df_{32} + f_{33}) = 0$

(33)

Subtract (2c)-(1c) and subtract (3c)-(1c) to obtain 2 equations in 2 unknowns,

$$\begin{bmatrix} r_{10} - r_{00} & u_{10} - u_{00} \\ r_{01} - r_{00} & u_{01} - u_{00} \end{bmatrix} \begin{bmatrix} f_{20} \\ f_{21} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{34}$$

The rows of the matrix are linearly independent, so the solution must be $f_{20} = 0$ and $f_{21} = 0$. Equation (1c) becomes $nf_{22} + f_{23} = 0$. These conditions eliminate equations (1c), (2c), (3c), and (4c). In summary we know that

$$f_{20} = 0,\ f_{21} = 0,\ f_{23} = -nf_{22} \tag{35}$$

The equations reduce to

label	equation
(1a)	$r_{00}f_{00} + u_{00}f_{01} + nf_{02} + f_{03} + (r_{00}f_{30} + u_{00}f_{31} + nf_{32} + f_{33}) = 0$
(1b)	$r_{00}f_{10} + u_{00}f_{11} + nf_{12} + f_{13} + (r_{00}f_{30} + u_{00}f_{31} + nf_{32} + f_{33}) = 0$
(2a)	$r_{10}f_{00} + u_{10}f_{01} + nf_{02} + f_{03} - (r_{10}f_{30} + u_{10}f_{31} + nf_{32} + f_{33}) = 0$
(2b)	$r_{10}f_{10} + u_{10}f_{11} + nf_{12} + f_{13} + (r_{10}f_{30} + u_{10}f_{31} + nf_{32} + f_{33}) = 0$
(3a)	$r_{01}f_{00} + u_{01}f_{01} + nf_{02} + f_{03} + (r_{01}f_{30} + u_{01}f_{31} + nf_{32} + f_{33}) = 0$
(3b)	$r_{01}f_{10} + u_{01}f_{11} + nf_{12} + f_{13} - (r_{01}f_{30} + u_{01}f_{31} + nf_{32} + f_{33}) = 0$
(4a)	$r_{11}f_{00} + u_{11}f_{01} + nf_{02} + f_{03} - (r_{11}f_{30} + u_{11}f_{31} + nf_{32} + f_{33}) = 0$
(4b)	$r_{11}f_{10} + u_{11}f_{11} + nf_{12} + f_{13} - (r_{11}f_{30} + u_{11}f_{31} + nf_{32} + f_{33}) = 0$
(5a)	$(r_{00}d/n)f_{00} + (u_{00}d/n)f_{01} + df_{02} + f_{03} + ((r_{00}d/n)f_{30} + (u_{00}d/n)f_{31} + df_{32} + f_{33}) = 0$
(5b)	$(r_{00}d/n)f_{10} + (u_{00}d/n)f_{11} + df_{12} + f_{13} + ((r_{00}d/n)f_{30} + (u_{00}d/n)f_{31} + df_{32} + f_{33}) = 0$
(5c)	$df_{22} + f_{23} - ((r_{00}d/n)f_{30} + (u_{00}d/n)f_{31} + df_{32} + f_{33}) = 0$
(6a)	$(r_{10}d/n)f_{00} + (u_{10}d/n)f_{01} + df_{02} + f_{03} - ((r_{10}d/n)f_{30} + (u_{10}d/n)f_{31} + df_{32} + f_{33}) = 0$
(6b)	$(r_{10}d/n)f_{10} + (u_{10}d/n)f_{11} + df_{12} + f_{13} + ((r_{10}d/n)f_{30} + (u_{10}d/n)f_{31} + df_{32} + f_{33}) = 0$
(6c)	$df_{22} + f_{23} - ((r_{10}d/n)f_{30} + (u_{10}d/n)f_{31} + df_{32} + f_{33}) = 0$
(7a)	$(r_{01}d/n)f_{00} + (u_{01}d/n)f_{01} + df_{02} + f_{03} + ((r_{01}d/n)f_{30} + (u_{01}d/n)f_{31} + df_{32} + f_{33}) = 0$
(7b)	$(r_{01}d/n)f_{10} + (u_{01}d/n)f_{11} + df_{12} + f_{13} - ((r_{01}d/n)f_{30} + (u_{01}d/n)f_{31} + df_{32} + f_{33}) = 0$
(7c)	$df_{22} + f_{23} - ((r_{01}d/n)f_{30} + (u_{01}d/n)f_{31} + df_{32} + f_{33}) = 0$
(8a)	$(r_{11}d/n)f_{00} + (u_{11}d/n)f_{01} + df_{02} + f_{03} - ((r_{11}d/n)f_{30} + (u_{11}d/n)f_{31} + df_{32} + f_{33}) = 0$
(8b)	$(r_{11}d/n)f_{10} + (u_{11}d/n)f_{11} + df_{12} + f_{13} - ((r_{11}d/n)f_{30} + (u_{11}d/n)f_{31} + df_{32} + f_{33}) = 0$
(8c)	$df_{22} + f_{23} - ((r_{11}d/n)f_{30} + (u_{11}d/n)f_{31} + df_{32} + f_{33}) = 0$

(36)

Subtract (6c)-(5c), subtract (7c)-(5c), and multiply the resulting equations by n/d to obtain 2 equations in 2 unknowns,

$$\begin{bmatrix} r_{10} - r_{00} & u_{10} - u_{00} \\ r_{01} - r_{00} & u_{01} - u_{00} \end{bmatrix} \begin{bmatrix} f_{30} \\ f_{31} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{37}$$

The rows of the matrix are linearly independent, so the solution must be $f_{30} = 0$ and $f_{31} = 0$. Equation (5c) becomes $(df_{22} + f_{23}) - (df_{32} + f_{33}) = 0$. These conditions eliminate equations (5c), (6c), (7c), and (8c). In summary we know that

$$\begin{aligned} &f_{20} = 0,\ f_{21} = 0,\ f_{23} = -nf_{22} \\ &f_{30} = 0,\ f_{31} = 0,\ (df_{22} + f_{23}) - (df_{32} + f_{33}) = 0 \end{aligned} \tag{38}$$

The equations reduce to the following, where (5a) through (8b) are multiplied by n/d to allow comparisons among the equations for the r_{ij}- and u_{ij}-terms,

label	equation
(1a)	$r_{00}f_{00} + u_{00}f_{01} + ((nf_{02} + f_{03}) + (nf_{32} + f_{33})) = 0$
(1b)	$r_{00}f_{10} + u_{00}f_{11} + ((nf_{12} + f_{13}) + (nf_{32} + f_{33})) = 0$
(2a)	$r_{10}f_{00} + u_{10}f_{01} + ((nf_{02} + f_{03}) - (nf_{32} + f_{33})) = 0$
(2b)	$r_{10}f_{10} + u_{10}f_{11} + ((nf_{12} + f_{13}) + (nf_{32} + f_{33})) = 0$
(3a)	$r_{01}f_{00} + u_{01}f_{01} + ((nf_{02} + f_{03}) + (nf_{32} + f_{33})) = 0$
(3b)	$r_{01}f_{10} + u_{01}f_{11} + ((nf_{12} + f_{13}) - (nf_{32} + f_{33})) = 0$
(4a)	$r_{11}f_{00} + u_{11}f_{01} + ((nf_{02} + f_{03}) - (nf_{32} + f_{33})) = 0$
(4b)	$r_{11}f_{10} + u_{11}f_{11} + ((nf_{12} + f_{13}) - (nf_{32} + f_{33})) = 0$
(5a)	$r_{00}f_{00} + u_{00}f_{01} + (n/d)((df_{02} + f_{03}) + (df_{32} + f_{33})) = 0$
(5b)	$r_{00}f_{10} + u_{00}f_{11} + (n/d)((df_{12} + f_{13}) + (df_{32} + f_{33})) = 0$
(6a)	$r_{10}f_{00} + u_{10}f_{01} + (n/d)((df_{02} + f_{03}) - (df_{32} + f_{33})) = 0$
(6b)	$r_{10}f_{10} + u_{10}f_{11} + (n/d)((df_{12} + f_{13}) + (df_{32} + f_{33})) = 0$
(7a)	$r_{01}f_{00} + u_{01}f_{01} + (n/d)((df_{02} + f_{03}) + (df_{32} + f_{33})) = 0$
(7b)	$r_{01}f_{10} + u_{01}f_{11} + (n/d)((df_{12} + f_{13}) - (df_{32} + f_{33})) = 0$
(8a)	$r_{11}f_{00} + u_{11}f_{01} + (n/d)((df_{02} + f_{03}) - (df_{32} + f_{33})) = 0$
(8b)	$r_{11}f_{10} + u_{11}f_{11} + (n/d)((df_{12} + f_{13}) - (df_{32} + f_{33})) = 0$

(39)

Subtract (5a)-(1a), subtract (5b)-(1b), subtract (6b)-(2a), and subtract (7b)-(3b) to obtain $(f_{03} + f_{33})(n - d)/d = 0$, $(f_{13} + f_{33})(n - d)/d = 0$, $(f_{03} - f_{33})(n - d)/d = 0$, and $(f_{13} - f_{33})(n - d)/d = 0$. These imply $f_{03} = 0$, $f_{13} = 0$, and $f_{33} = 0$. In summary we know that the following hold (with some algebra),

$$\begin{aligned}
&f_{20} = 0,\ f_{21} = 0,\ f_{23} = -nf_{22} \\
&f_{30} = 0,\ f_{31} = 0,\ f_{32} = ((d - n)/d)f_{22} \\
&f_{03} = 0,\ f_{13} = 0,\ f_{33} = 0
\end{aligned} \tag{40}$$

The equations reduce to

label	equation
(1a)	$r_{00}f_{00} + u_{00}f_{01} + n(f_{02} + f_{32}) = 0$
(1b)	$r_{00}f_{10} + u_{00}f_{11} + n(f_{12} + f_{32}) = 0$
(2a)	$r_{10}f_{00} + u_{10}f_{01} + n(f_{02} - f_{32}) = 0$
(2b)	$r_{10}f_{10} + u_{10}f_{11} + n(f_{12} + f_{32}) = 0$
(3a)	$r_{01}f_{00} + u_{01}f_{01} + n(f_{02} + f_{32}) = 0$
(3b)	$r_{01}f_{10} + u_{01}f_{11} + n(f_{12} - f_{32}) = 0$
(4a)	$r_{11}f_{00} + u_{11}f_{01} + n(f_{02} - f_{32}) = 0$
(4b)	$r_{11}f_{10} + u_{11}f_{11} + n(f_{12} - f_{32}) = 0$
(5a)	$r_{00}f_{00} + u_{00}f_{01} + n(f_{02} + f_{32}) = 0$
(5b)	$r_{00}f_{10} + u_{00}f_{11} + n(f_{12} + f_{32}) = 0$
(6a)	$r_{10}f_{00} + u_{10}f_{01} + n(f_{02} - f_{32}) = 0$
(6b)	$r_{10}f_{10} + u_{10}f_{11} + n(f_{12} + f_{32}) = 0$
(7a)	$r_{01}f_{00} + u_{01}f_{01} + n(f_{02} + f_{32}) = 0$
(7b)	$r_{01}f_{10} + u_{01}f_{11} + n(f_{12} - f_{32}) = 0$
(8a)	$r_{11}f_{00} + u_{11}f_{01} + n(f_{02} - f_{32}) = 0$
(8b)	$r_{11}f_{10} + u_{11}f_{11} + n(f_{12} - f_{32}) = 0$

(41)

Observe that (1*) and (5*) are the same, (2*) and (6*) are the same, (3*) and (7*) are the same, and (4*) and (8*) are the same. The relevant equations are now

label	equation
(1a)	$r_{00}f_{00} + u_{00}f_{01} + n(f_{02} + f_{32}) = 0$
(1b)	$r_{00}f_{10} + u_{00}f_{11} + n(f_{12} + f_{32}) = 0$
(2a)	$r_{10}f_{00} + u_{10}f_{01} + n(f_{02} - f_{32}) = 0$
(2b)	$r_{10}f_{10} + u_{10}f_{11} + n(f_{12} + f_{32}) = 0$
(3a)	$r_{01}f_{00} + u_{01}f_{01} + n(f_{02} + f_{32}) = 0$
(3b)	$r_{01}f_{10} + u_{01}f_{11} + n(f_{12} - f_{32}) = 0$
(4a)	$r_{11}f_{00} + u_{11}f_{01} + n(f_{02} - f_{32}) = 0$
(4b)	$r_{11}f_{10} + u_{11}f_{11} + n(f_{12} - f_{32}) = 0$

(42)

The coefficient f_{32} depends on f_{22}. We have 8 equations in 7 unknowns f_{00}, f_{01}, f_{02}, f_{10}, f_{11}, f_{12}, and f_{22}. As in the case of a general view frustum, it appears that f_{22} can be a free parameter, so we have 8 equations in 6 unknowns.

Subtract (2a)-(1a), subtract (3a)-(1a), subtract (2b)-(1b), and subtract (3b)-(1b) to obtain a

$$\begin{aligned}
(r_{10} - r_{00})f_{00} + (u_{10} - u_{00})f_{01} &= c \\
(r_{01} - r_{00})f_{00} + (u_{01} - u_{00})f_{01} &= 0 \\
(r_{10} - r_{00})f_{10} + (u_{11} - u_{00})f_{11} &= 0 \\
(r_{01} - r_{00})f_{10} + (u_{01} - u_{00})f_{11} &= c
\end{aligned} \tag{43}$$

where $c = (2n(d - n)/d)f_{22} \neq 0$. As a matrix system we have

$$MF = \begin{bmatrix} r_{10} - r_{00} & u_{10} - u_{00} \\ r_{01} - r_{00} & u_{01} - u_{00} \end{bmatrix} \begin{bmatrix} f_{00} & f_{10} \\ f_{01} & f_{11} \end{bmatrix} = \begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix} = cI \tag{44}$$

where I is the 2×2 identity matrix. The rows of M are linearly independent, so M is invertible. The solution is $F = cM^{-1}$.

The nonexistence of a perspective mapping is based on the final analysis of (4a) and (4b). The linear independence of the rows of M allow us to represent $(r_{11} - r_{00}, u_{11} - u_{00}) = a(r_{10} - r_{00}, u_{10} - u_{10}) + b(r_{01} - r_{00}, u_{01} - u_{00})$ for coefficients $a > 0$, $b > 0$, and $a + b \geq 1$. The conditions for the coefficients is based on the viewport being a convex quadrilateral. Substituting the representation into (4a) and (4b) and regrouping terms leads to

$$\begin{aligned}
a((r_{10} - r_{00})f_{00} + (u_{10} - u_{10})f_{01}) + b((r_{01} - r_{00})f_{00} + (u_{01} - u_{00})f_{01}) &= c \\
a((r_{10} - r_{00})f_{10} + (u_{10} - u_{10})f_{11}) + b((r_{01} - r_{00})f_{10} + (u_{01} - u_{00})f_{11}) &= c
\end{aligned} \tag{45}$$

Using equation (43), we obtain $(a(c) + b(0) = c)$ and $(a(0) + b(c) = c)$. Given that c is not zero, the only possible conclusion is that $a = 1$ and $b = 1$, which forces the convex quadrilateral viewport to be a parallelogram.

References

[1] Wolfram Research, Inc. *Mathematica 14.0.0*. Wolfram Research, Inc., Champaign, Illinois, 2024.