Parallel Projection of an Ellipse

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1 Discussion

This document describes an algorithm for projecting an ellipse in three dimensions onto a plane using a parallel projection.

The plane of the ellipse is $\mathbf{N}_e \cdot (\mathbf{X}_e - \mathbf{C}_e) = 0$, where $\mathbf{N}_e$ is a unit-length normal vector, $\mathbf{C}_e$ is the plane origin, and $\mathbf{X}_e$ is any point on the plane. Let $\mathbf{U}_e$ and $\mathbf{V}_e$ be vectors such that $\{\mathbf{U}_e, \mathbf{V}_e, \mathbf{N}_e\}$ is an orthonormal set with $\mathbf{N}_e = \mathbf{U}_e \times \mathbf{V}_e$. Points on the plane are represented by

$$\mathbf{X}_e = \mathbf{C}_e + y_{e,0}\mathbf{U}_e + y_{e,1}\mathbf{V}_e = \mathbf{C}_e + \left[\begin{array}{c|c} \mathbf{U}_e & \mathbf{V}_e \end{array} \right] \left[\begin{array}{c} y_{e,0} \\ y_{e,1} \end{array} \right] = \mathbf{C}_e + J_e \mathbf{Y}_e \tag{1}$$

where J_e is the 3×2 matrix whose columns are the 3×1 vectors $\mathbf{U}_e$ and $\mathbf{V}_e$ and where $\mathbf{Y}_e$ is the 2×1 vector whose rows are $y_{e,0}$ and $y_{e,1}$. The use of subscript e is to remind you these quantities are associated with the ellipse plane.

Similarly, the projection plane is $\mathbf{N}_p \cdot (\mathbf{X}_p - \mathbf{C}_p) = 0$, where $\mathbf{N}_p$ is a unit-length normal vector, $\mathbf{C}_p$ is the plane origin, and $\mathbf{X}_p$ is any point on the plane. Let $\mathbf{U}_p$ and $\mathbf{V}_p$ be vectors such that $\{\mathbf{U}_p, \mathbf{V}_p, \mathbf{N}_p\}$ is an orthonormal set with $\mathbf{N}_p = \mathbf{U}_p \times \mathbf{V}_p$. Points on the plane are represented by

$$\mathbf{X}_p = \mathbf{C}_p + y_{p,0}\mathbf{U}_p + y_{p,1}\mathbf{V}_p = \mathbf{C}_p + \left[\begin{array}{c|c} \mathbf{U}_p & \mathbf{V}_p \end{array} \right] \left[\begin{array}{c} y_{p,0} \\ y_{p,1} \end{array} \right] = \mathbf{C}_p + J_p \mathbf{Y}_p \tag{2}$$

where J_p is the 3×2 matrix whose columns are the 3×1 vectors $\mathbf{U}_p$ and $\mathbf{V}_p$ and where $\mathbf{Y}_p$ is the 2×1 vector whose rows are $y_{p,0}$ and $y_{p,1}$. The use of subscript p is to remind you these quantities are associated with the projection plane.

The projection is parallel to a specified unit-length direction vector $\mathbf{D}$. For the projection to be invertible, it is required that $\mathbf{N}_p \cdot \mathbf{D} \neq 0$ and $\mathbf{N}_e \cdot \mathbf{D} \neq 0$. The first condition guarantees the projection actually exists and is *onto* the projection plane. The second condition guarantees that the projection is *one-to-one* (the ellipse does not project to a line segment). The two conditions, onto and one-to-one, mean the projection is invertible. The projection of $\mathbf{X}_e$ to a point $\mathbf{X}_p$ is the intersection of the line $\mathbf{X}_p = \mathbf{X}_e + t\mathbf{D}$ with the projection plane. Substituting this into the projection plane equation $\mathbf{N}_p \cdot (\mathbf{X}_p - \mathbf{C}_p) = 0$ and solving for t produces the intersection point

$$\mathbf{X}_p = \mathbf{X}_e - \frac{\mathbf{N}_p \cdot (\mathbf{X}_e - \mathbf{C}_p)}{\mathbf{D} \cdot \mathbf{N}_p} \mathbf{D} \tag{3}$$

The ellipse plane origin was chosen to be the ellipse center $\mathbf{C}_e$. We have the freedom to choose the projection plane origin any way we like. Conveniently, let us choose it to be the projection of the ellipse center onto the projection plane, specifically,

$$\mathbf{C}_p = \mathbf{C}_e - \frac{\mathbf{N}_p \cdot (\mathbf{C}_e - \mathbf{C}_p)}{\mathbf{D} \cdot \mathbf{N}_p} \mathbf{D} \tag{4}$$

Consequently, equation (3) is equivalent to

$$\mathbf{X}_p - \mathbf{C}_p = \mathbf{X}_e - \mathbf{C}_e - \frac{\mathbf{N}_p \cdot (\mathbf{X}_e - \mathbf{C}_e)}{\mathbf{D} \cdot \mathbf{N}_p} \mathbf{D} = \left(I_3 - \frac{\mathbf{D}\mathbf{N}_p^{\mathsf{T}}}{\mathbf{D} \cdot \mathbf{N}_p} \right) (\mathbf{X}_e - \mathbf{C}_e) \tag{5}$$

where I_3 is the 3×3 identity matrix.

The plane origins in equations (1) and (2) may be subtracted from the equations to obtain $\mathbf{X}_e - \mathbf{C}_e = J_e \mathbf{Y}_e$ and $\mathbf{X}_p - \mathbf{C}_p = J_p \mathbf{Y}_p$. Substitute these into equation (5) to obtain

$$J_p \mathbf{Y}_p = \left(I_3 - \frac{\mathbf{D}\mathbf{N}_p^{\mathsf{T}}}{\mathbf{D} \cdot \mathbf{N}_p} \right) J_e \mathbf{Y}_e \tag{6}$$

Using the fact that $J_p^{\mathsf{T}} J_p = I_2$, where I_2 is the 2×2 identity matrix, we have

$$\mathbf{Y}_p = J_p^{\mathsf{T}} \left(I_3 - \frac{\mathbf{D}\mathbf{N}_p^{\mathsf{T}}}{\mathbf{D} \cdot \mathbf{N}_p} \right) J_e \mathbf{Y}_e = A \mathbf{Y}_e \tag{7}$$

where the second equality defines the 2×2 matrix A. Since we know the projection is invertible, it must be the case that A is invertible, in which case

$$\mathbf{Y}_e = A^{-1} \mathbf{Y}_p \tag{8}$$

The ellipse is represented parametrically by

$$\mathbf{X}_e(\theta) = \mathbf{C}_e + (a \cos\theta)\mathbf{U}_e + (b \sin\theta)\mathbf{V}_e = \mathbf{C}_e + J_e \mathbf{Y}_e(\theta) \tag{9}$$

where $\mathbf{Y}_e(\theta) = [a \cos\theta \; b \sin\theta]^{\mathsf{T}}$, a 2×1 vector, and where $\theta \in [0, 2\pi)$. The positive values a and b are the ellipse major and minor axis lengths. Define Δ to be the diagonal matrix whose diagonal entries are $1/a$ and $1/b$. Then

$$\Delta \mathbf{Y}_e = \begin{bmatrix} 1/a & 0 \\ 0 & 1/b \end{bmatrix} \begin{bmatrix} a \cos\theta \\ b \sin\theta \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

Since the right-hand side is a unit-length vector, we see that

$$\mathbf{Y}_e^{\mathsf{T}} \Delta^2 \mathbf{Y}_e = 1 \tag{10}$$

Substituting equation (8) into this produces

$$\mathbf{Y}_p^{\mathsf{T}} M \mathbf{Y}_p = \mathbf{Y}_p^{\mathsf{T}} \left(A^{-\mathsf{T}} \Delta^2 A^{-1} \right) \mathbf{Y}_p = 1 \tag{11}$$

where $A^{-\mathsf{T}}$ is the transpose of the inverse of A. The first equality defines the matrix M. Equation (11) is the representation of the projected ellipse in the coordinate system of the projection plane. The projection is itself an ellipse. The matrix M can be factored into $M = RDR^{\mathsf{T}}$, where R is a rotation matrix whose columns are eigenvectors for M and where $D = \text{Diag}(d_0, d_1)$ are the eigenvalues, both positive. The projected ellipse major and minor axis lengths are $1/\sqrt{d_0}$ and $1/\sqrt{d_1}$.