Representing a Circle or a Sphere with NURBS

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Created: January 13, 2003
Last Modified: September 11, 2020

Contents

This document describes representing circles and spheres with NURBS. For more information about NURBS, a good engineering-style approach to NURBS is [3]. An excellent book covering the mathematics and modeling with splines is [1]. A more mathematically advanced presentation is [2].

1 Introduction

A NURBS curve in 2D is generated by a homogeneous B-spline curve in 3D,

$$(\bar{x}(u), \bar{y}(u), w(u)) = \sum_{i=0}^{n} N_{i,d}(u) w_i (\mathbf{C}_i, 1) \tag{1}$$

where the 3D curve has $n+1$ control points $(\mathbf{C}_i, 1)$ [the $\mathbf{C}_i$ are 2-tuples], $n+1$ weights w_i, and degree d with $1 \leq d \leq n$. The functions $N_{i,d}(u)$ are the B-spline basis functions, which are defined recursively and require selection of a sequence of nondecreasing scalars u_i for $0 \leq i \leq n+d+1$. Each u_i is called a knot and the total sequence is called a knot vector. The basis function that starts the recursive definition is

$$N_{i,0}(u) = \begin{cases} 1, & u_i \leq u < u_{i+1} \\ 0, & \text{otherwise} \end{cases} \tag{2}$$

for $0 \leq i \leq n+d$. The recursion is

$$N_{i,j}(u) = \frac{u - u_i}{u_{i+j} - u_i} N_{i,j-1}(u) + \frac{u_{i+j+1} - u}{u_{i+j+1} - u_{i+1}} N_{i+1,j-1}(u) \tag{3}$$

for $1 \leq j \leq d$ and $0 \leq i \leq n+d-j$. Equation (1) uses only the final evaluations in the recursion, $N_{i,d}(u)$ for which $j = d$ and $0 \leq i \leq n$. The 2D curve is obtained by the perspective division,

$$(x(u), y(u)) = (\bar{x}(u), \bar{y}(u))/w(u) \tag{4}$$

In the section on circle representations, I will discuss NURBS curves that produce circular arcs; each curve will be listed with its n, d, u_i, $\mathbf{C}_i$, and w_i.

A tensor-product NURBS surface in 3D is generated by a homogeneous B-spline surface in 4D,

$$(\bar{x}(u,v), \bar{y}(u,v), \bar{z}(u,v), w(u,v)) = \sum_{i_0=0}^{n_0} \sum_{i_1=0}^{n_1} N_{i_0,d_0}(u) N_{i_1,d_1}(v) w_{i_0 i_1} (\mathbf{C}_{i_0 i_1}, 1) \tag{5}$$

where the 4D surface has an $(n_0+1) \times (n_1+1)$ array of control points $(\mathbf{C}_{i_0 i_1}, 1)$ [the $\mathbf{C}_{i_0 i_1}$ are 3-tuples], an $(n_0+1) \times (n_1+1)$ array of weights $w_{i_0 i_1}$, and degrees d_0 and d_1 with $1 \leq d_0 \leq n_0$ and $1 \leq d_1 \leq n_1$. The functions $N_{i_0,d_0}(u)$ and $N_{i_1,d_1}(v)$ each have a knot vector and are B-spline basis functions generated by equations (2) and (3). The 3D surface is obtained by the perspective division,

$$(x(u,v), y(u,v), z(u,v)) = (\bar{x}(u,v), \bar{y}(u,v), \bar{z}(u,v))/w(u,v) \tag{6}$$

In the section on sphere representations, I will discuss NURBS surfaces that produce spherical regions; each surface will be listed with its n_0, n_1, d_0, d_1, u_i, v_i, $\mathbf{C}_{i_0 i_1}$, and $w_{i_0 i_1}$.

2 Representing a Circle

Several representations are provided for a quarter of a circle in the first quadrant. A representation is also provided for half a circle. All these representation use the same knot pattern involving only the repeated zeros and ones at the beginning and end of the knot array. Finally, a representation for a full circle is created from that of the half circle by introducing a repeated knot of one-half that has the effect of splicing together two regular NURBS surfaces.

2.1 Quarter Circle

Consider the quarter circle in the first quadrant, $x^2 + y^2 = 1$ with $x \geq 0$ and $y \geq 0$.

2.1.1 Degree 2

A quadrant of a circle can be represented as a NURBS curve of degree 2. The curve is $x^2 + y^2 = 1$, $x \geq 0$, $y \geq 0$. The general parameterization is

$$(\bar{x}(u), \bar{y}(u), w(u)) = w_0(1-u)^2(1,0,1) + w_1 2u(1-u)(1,1,1) + w_2 u^2(0,1,1) \tag{7}$$

for $u \in [0,1]$. The parameterization is a ratio of quadratic polynomials, $(x(u), y(u)) = (\bar{x}(u)/w(u), \bar{y}(u)/w(u))$. The requirement that $x(u)^2 + y^2(u) = 1$ leads to the constraint $2w_1^2 = w_0 w_2$.

The choice of weights $w_0 = 1$, $w_1 = 1$, and $w_2 = 2$ leads to the parameterization

$$(x(u), y(u)) = \frac{(1-u^2, 2u)}{1+u^2} \tag{8}$$

If you were to tessellate the curve with an odd number of uniform samples of u, say $u_i = i/(2n)$ for $0 \leq i \leq 2n$, then the resulting polyline is not symmetric about the midpoint $u = 1/2$.

To obtain a symmetric tessellation you need to choose $w_0 = w_2$. The weight constraint then implies $w_0 = w_1\sqrt{2}$. The parameterization is then

$$(x(u), y(u)) = \frac{(\sqrt{2}(1-u)^2 + 2u(1-u), 2u(1-u) + \sqrt{2}u^2)}{\sqrt{2}(1-u)^2 + 2u(1-u) + \sqrt{2}u^2} \tag{9}$$

The NURBS parameters are the following. There are 3 control points, so $n = 2$. The degree is $d = 2$. The knot vector is $\{u_0, u_1, u_2, u_3, u_4, u_5\} = \{0,0,0,1,1,1\}$. The control values are $\mathbf{C}_0 = (1,0)$, $\mathbf{C}_1 = (1,1)$, and $\mathbf{C}_2 = (0,1)$. The weights are w_0, w_1, and w_2 subject to the constraint $2w_1^2 = w_0 w_2$.

2.1.2 Degree 4

An algebraic construction of the same type produces a ratio of quartic polynomials. The control points and control weights are required to be symmetric to obtain a tessellation that is symmetric about its midpoint. Define $P_{d,k}(u) = (1-u)^{d-k}u^k$ for $d > 0$ and $0 \leq k \leq d$. The NURBS curve is

$$\begin{aligned}(\bar{x}(u), \bar{y}(u), w(u)) \quad &= \quad P_{4,0}(u)w_0(1,0,1) + 4P_{4,1}(u)w_1(x_1,y_1,1) + 6P_{4,2}(u)w_2(x_2,x_2,1) \\ &\quad + 4P_{4,3}(u)w_1(y_1,x_1,1) + P_{4,4}(u)w_0(0,1,1)\end{aligned} \tag{10}$$

The parameterization is a ratio of quartic polynomials, $(x(u), y(u)) = (\bar{x}(u)/w(u), \bar{y}(u)/w(u))$. We have one degree of freedom, choosing $w_0 = 1$ which effectively divides the numerators and denominators by w_0.

To be a circle, we need $x(u)^2 + y(u)^2 = 1$, which is equivalent to $\bar{x}^2(u) + \bar{y}^2(u) - w^2(u) = 0$. Some algebraic manipulation leads to

$$
\begin{aligned}
0 \quad = \quad & [P_{8,1}(u) + P_{8,7}(u)][8w_1(x_1 - 1)] + \\
& [P_{8,2}(u) + P_{8,6}(u)][12w_2(x_2 - 1) + 16w_1^2(x_1^2 + y_1^2 - 1)] + \\
& [P_{8,3}(u) + P_{8,5}(u)][8w_1(y_1 - 1) + 48w_1w_2(x_2(x_1 + y_1) - 1)] + \\
& [P_{8,4}(u) + P_{8,4}(u)][16w_1^2(2x_1y_1 - 1) + 18w_2^2(2x_2^2 - 1) - 1]
\end{aligned}
\tag{11}
$$

The four u-polynomials are linearly independent in the vector space of u-polynomials, which means their four coefficients must all be zero,

$$
\begin{aligned}
& w_1(x_1 - 1) = 0 \\
& 3w_2(x_2 - 1) + 4w_1^2(x_1^2 + y_1^2 - 1) = 0 \\
& w_1(y_1 - 1) + 6w_1w_2(x_2(x_1 + y_1) - 1) = 0 \\
& 16w_1^2(2x_1y_1 - 1) + 18w_2^2(2x_2^2 - 1) - 1 = 0
\end{aligned}
\tag{12}
$$

The solution space is constrained to $w_1 \geq 0$, $w_2 \geq 0$, $x_1 \geq 0$, $y_1 \geq 0$ and $x_2 \geq 0$. It cannot be that $w_2 = 0$; for if it were, the fourth equation requires $w_1 \neq 0$. The first equation implies $x_1 = 1$ and the third equation implies $y_1 = 1$. The second equation then implies $w_1 = 0$, which is a contradiction. The conclusion is that $w_2 > 0$.

If $w_1 = 0$, then $x_2 = 1$ and $w_2 = 1/\sqrt{18}$. The values of x_1 and y_1 are irrelevant, because $w_1 = 0$ annihilates the $(x_1, y_1, 1)$ and $(y_1, x_1, 1)$ terms in equation (10), leading to the parameterization

$$
(\bar{x}(u), \bar{y}(u), w(u)) = (1-u)^4(1, 0, 1) + \sqrt{2}(1-u)^2u^2(1, 1, 1) + u^4(0, 1, 1)
\tag{13}
$$

A plot of the parameterized curve $(\bar{x}(u)/w(u)), \bar{y}(u)/w(u))$ shows that the curve is indeed a quarter circle. Multiplying numerators and denominator of the parameterized curve by $\sqrt{2}$ leads to a parameterization similar to that of equation (9).

If $w_1 \neq 0$, then $x_1 = 1$. Using Mathematics [4], the equations (12) were solved with free variable w_2 to produce

$$
w_1 = \frac{3w_2}{2\sqrt{2}}, \quad x_2 = 1 - \frac{1}{6w_2}, \quad y_1 = \frac{1}{3w_2}
\tag{14}
$$

or

$$
w_1 = \frac{1}{2}\sqrt{\frac{1}{2} + \left(1 + \frac{1}{\sqrt{2}}\right)(3w_2 - 1)}, \quad x_2 = \frac{(3w_2 + 1)\sqrt{2} - 1}{6w_2}, \quad y_1 = \sqrt{2} - 1
\tag{15}
$$

where $w_2 \geq 1/3$.

Thus, we have at least one more degree of freedom and there are infinitely many solutions. For example, if we choose $w_1 = 1$, then there are 2 possibilities for the remaining parameters: $(y_1, x_2, w_2) = (1/(2\sqrt{2}), 1 - \sqrt{2}/8, 2\sqrt{2}/3)$ and $(y_1, x_2, w_2) = (\sqrt{2} - 1, (44\sqrt{2} - 1)/79, (16 - 7\sqrt{2})/6)$. With these choices, Mathematica verified that $x(u)^2 + y(u)^2 = 1$ and parametric plots of $(x(u), y(u))$ were indeed quarter circles. Mathematica also symbolically generated solutions to the constraints when asked to solve for y_1, x_2, and w_2 in terms of w_1.

The NURBS parameters are the following. There are 5 control points, so $n = 4$. The degree is $d = 4$. The knot vector has 10 components with $u_i = 0$ for $0 \le i \le 4$ and $u_i = 1$ for $5 \le i \le 9$. Assuming the constraints of equation (12) are satisfied, the control values are $\mathbf{C}_0 = (1,0)$, $\mathbf{C}_1 = (x_1, y_1)$, $\mathbf{C}_2 = (x_2, x_2)$, $\mathbf{C}_3 = (y_1, x_1)$, and $\mathbf{C}_4 = (0,1)$ and the weights are $w_0 = w_4 = 1$, w_2, and $w_1 = w_3$.

2.2 Half Circle

This section describes a symmetric parameterization of the half circle $x^2 + y^2 = 1$ for $y \ge 0$. The NURBS curve has degree 3. The control values are chosen to be $(1,0)$, $(1,\alpha)$, $(-1,\alpha)$ and $(-1,0)$ where $\alpha > 0$ and the weights are chosen to be $w_0 = w_3 = 1$ and $w_1 = w_2 = \omega > 0$. The parameters α and ω must be chosen so that we indeed obtain a half circle. The curve is

$$(x(u), y(u)) = \frac{(1-u)^3(1,0) + 3u(1-u)^2\omega(1,\alpha) + 3u^2(1-u)\omega(-1,\alpha) + u^3(-1,0)}{(1-u)^3 + 3u(1-u)^2\omega + 3u^2(1-u)\omega + u^3} \tag{16}$$

Symbolic manipulation of $x(u)^2 + y(u)^2 = 1$ leads to the constraints $\omega = 1/3$ and $\alpha = 2$.

The NURBS parameters are the following. There are 4 control points, so $n = 3$. The degree is $d = 3$. The knot vector has 8 components with $u_i = 0$ for $0 \le i \le 3$ and $u_i = 1$ for $4 \le i \le 7$. The control values are $\mathbf{C}_0 = (1,0)$, $\mathbf{C}_1 = (1,2)$, $\mathbf{C}_2 = (-1,2)$, and $\mathbf{C}_3 = (-1,0)$. The weights are $w_0 = w_3 = 1$ and $w_1 = w_2 = 1/3$.

2.3 Full Circle

The full-circle curve can be obtained by splicing together two half-circle curves as constructed in the previous section ($d = 3$). The 7 control values ($n = 6$) are in the counterclockwise order $(1,0)$, $(1,2)$, $(-1,2)$, $(-1,0)$, $(-1,-2)$, $(1,-2)$, and $(1,0)$. Observe that the first and last controls are duplicated in order to close the curve. The splicing occurs at the control value $(0,-1)$. Rather than duplicate the control, the splicing is accomplished by choosing an interior knot of $1/2$ with multiplicity 3; the knots are $u_i = 0$ for $0 \le i \le 3$, $u_i = 1/2$ for $4 \le i \le 6$, and $u_i = 1$ for $7 \le i \le 10$. Note that the number of knots is as required: $n+d+2 = 11$.

Equation (2) provides the initial functions for starting the recursion. These are $N_{i,0}(u) = 0$ for $i \in \{0,1,2,4,5,7,8,9\}$ and

$$\begin{aligned} N_{3,0}(u) &= \{1, \quad u \in [0,1/2); \quad 0, \quad u \in [1/2,1)\} \\ N_{6,0}(u) &= \{0, \quad u \in [0,1/2); \quad 1, \quad u \in [1/2,1)\} \end{aligned} \tag{17}$$

For $j = 1$, equation (3) produces $N_{i,1}(u) = 0$ for $i \in \{0,1,4,7,8\}$ and

$$\begin{array}{lcllll} N_{2,1}(u) & = & \{1-2u, & u \in [0,1/2); & 0, & u \in [1/2,1)\} \\ N_{3,1}(u) & = & \{2u, & u \in [0,1/2); & 0, & u \in [1/2,1)\} \\ N_{5,1}(u) & = & \{0, & u \in [0,1/2); & 2-2u, & u \in [1/2,1)\} \\ N_{6,1}(u) & = & \{0, & u \in [0,1/2); & 2u-1, & u \in [1/2,1)\} \end{array} \tag{18}$$

For $j = 2$, equation (3) produces $N_{2,2}(u) = 0$, $N_{7,2}(u) = 0$, and

$$\begin{array}{rclllll} N_{1,2}(u) & = & \{(1-2u)^2, & u \in [0,1/2); & 0, & u \in [1/2,1)\} \\ N_{2,2}(u) & = & \{2(2u)(1-2u), & u \in [0,1/2); & 0, & u \in [1/2,1)\} \\ N_{3,2}(u) & = & \{(2u)^2, & u \in [0,1/2); & 0, & u \in [1/2,1)\} \\ N_{4,2}(u) & = & \{0, & u \in [0,1/2); & (2-2u)^2, & u \in [1/2,1)\} \\ N_{5,2}(u) & = & \{0, & u \in [0,1/2); & 2(2u-1)(2-2u), & u \in [1/2,1)\} \\ N_{6,2}(u) & = & \{0, & u \in [0,1/2); & (2u-1)^2, & u \in [1/2,1)\} \end{array} \tag{19}$$

For $j = 3$, equation (3) produces

$$\begin{array}{rclllll} N_{0,3}(u) & = & \{(1-2u)^3, & u \in [0,1/2); & 0, & u \in [1/2,1)\} \\ N_{1,3}(u) & = & \{3(2u)(1-2u)^2, & u \in [0,1/2); & 0, & u \in [1/2,1)\} \\ N_{2,3}(u) & = & \{3(2u)^2(1-2u), & u \in [0,1/2); & 0, & u \in [1/2,1)\} \\ N_{3,3}(u) & = & \{(2u)^3, & u \in [0,1/2); & (2-2u)^3, & u \in [1/2,1)\} \\ N_{4,3}(u) & = & \{0, & u \in [0,1/2); & 3(2u-1)(2-2u)^2, & u \in [1/2,1)\} \\ N_{5,3}(u) & = & \{0, & u \in [0,1/2); & 3(2u-1)^2(2-2u), & u \in [1/2,1)\} \\ N_{6,3}(u) & = & \{0, & u \in [0,1/2); & (2u-1)^3, & u \in [1/2,1)\} \end{array} \tag{20}$$

The last set of functions, $N_{i,3}(u)$, are what are used in the NURBS curve definition. Observe that the only function that has nonzero expressions for both $u \in [0, 1/2)$ and $u \in [1/2, 1)$ is $N_{3,3}(u)$. This is where the splicing occurs. The weights are the same as in the half-circle case. Table 1 summarizes the control points and weights.

Table 1. The control points, weights, and intervals of the domain for the circle.

controls	$(1,0)$	$(1,2)$	$(-1,2)$	$(-1,0)$	$(-1,-2)$	$(1,-2)$	$(1,0)$
weights	1	1/3	1/3	1	1/3	1/3	1
$u \in [0,1/2)$	$(1-2u)^3$	$3(2u)(1-2u)^2$	$3(2u)^2(1-2u)$	$(2u)^3$	0	0	0
$u \in [1/2,1)$	0	0	0	$(2-2u)^3$	$3(2u-1)(2-2u)^2$	$3(2u-1)^2(2-2u)$	$(2u-1)^3$

For $u \in [0, 1/2)$, the NURBS curve has nonzero weights only for the first four control points. For $u \in [1/2, 1)$, the NURBS curve has nonzero weights only for the last four control points.

3 Representing a Sphere

3.1 One Octant of a Sphere

3.1.1 Nonsymmetric, Degree 4

An octant of a sphere can be represented as a triangular NURBS surface patch of degree 4. A simple parameterization of $x^2 + y^2 + z^2 = 1$ can be made by setting $r^2 = x^2 + y^2$. The sphere is then $r^2 + z^2 = 1$. Now apply the parameterization for a circle,

$$(r, z) = \frac{(1 - u^2, 2u)}{1 + u^2} \tag{21}$$

But $(x/r)^2 + (y/r)^2 = 1$, so another application of the parameterization for a circle is

$$\frac{(x, y)}{r} = \frac{(1 - v^2, 2v)}{1 + v^2} \tag{22}$$

Combining these produces

$$(x(u, v), y(u, v), z(u, v)) = \frac{((1 - u^2)(1 - v^2), (1 - u^2)2v, 2u(1 + v^2))}{(1 + u^2)(1 + v^2)} \tag{23}$$

The components are ratios of quartic polynomials. The domain is $u \geq 0$, $v \geq 0$, and $u+v \leq 1$. In barycentric coordinates, set $w = 1 - u - v$ so that $u + v + w = 1$ with u, v, and w nonnegative. In this setting, you can think of the octant of the sphere as a mapping from the uvw-triangle with vertices $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$. Although a valid parameterization, a symmetric subdivision of the the uvw-triangle does not lead to a symmetric tessellation of the sphere.

3.1.2 Symmetric, Degree 4

Another parameterization is provided in [2]. This one chooses symmetric control points and symmetric weights,

$$(x(u, v), y(u, v), z(u, v)) = \frac{\sum_{i=0}^{4} \sum_{j=0}^{4-i} w_{i,j,4-i-j} \mathbf{P}_{i,j,4-i-j} B_{i,j}(u, v)}{\sum_{i=0}^{4} \sum_{j=0}^{4-i} w_{i,j,4-i-j} B_{i,j}(u, v)} \tag{24}$$

where

$$B_{i,j}(u, v) = \frac{4!}{i!j!(4 - i - j)!} u^i v^j t^{4-i-j}, \quad u \geq 0, \quad v \geq 0, \quad t = 1 - u - v \geq 0 \tag{25}$$

are the Bernstein polynomials. These can be written in a triangular form,

$$\begin{array}{lllll} v^4 & & & & \\ 4v^3 t & 4uv^3 & & & \\ 6v^2 t^2 & 12uv^2 t & 6u^2 v^2 & & \\ 4vt^3 & 12uvt^2 & 12u^2 vt & 4u^3 v & \\ t^4 & 4ut^3 & 6u^2 t^2 & 4u^3 t & u^4 \end{array} \tag{26}$$

In the implementation I allow for derivative computation through order 2. The partial derivatives $\partial B/\partial u$ are

$$\begin{array}{lllll} 0 & & & & \\ -4v^3 & 4v^3 & & & \\ -12v^2t & 12v^2(t-u) & 12uv^2 & & \\ -12vt^2 & 12vt(t-2u) & 12uv(2t-u) & 12u^2v & \\ -4t^3 & 4t^2(t-3u) & 12ut(t-u) & 4u^2(3t-u) & 4u^3 \end{array} \tag{27}$$

The partial derivatives $\partial B/\partial v$ are

$$\begin{array}{lllll} 4v^3 & & & & \\ 4v^2(3t-v) & 12uv^2 & & & \\ 12vt(t-v) & 12uv(2t-v) & 12u^2v & & \\ 4t^2(t-3v) & 12ut(t-2v) & 12u^2(t-v) & 4u^3 & \\ -t^3 & -12ut^2 & -12u^2t & -4u^3 & 0 \end{array} \tag{28}$$

The partial derivatives $\partial^2 B/\partial u^2$ are

$$\begin{array}{lllll} 0 & & & & \\ 0 & 0 & & & \\ 12v^2 & -24v^2 & 12v^2 & & \\ 24vt & -24v(2t-u) & 24v(t-2u) & 24uv & \\ 12t^2 & -24t(t-u) & 12(t^2-4ut+u^2) & 24u(t-u) & 12u^2 \end{array} \tag{29}$$

The partial derivatives $\partial^2 B/\partial u \partial v$ are

$$\begin{array}{lllll} 0 & & & & \\ -12v^2 & 12v^2 & & & \\ -12v(2t-v) & 12v(2t-2u-v) & 24uv & & \\ -12t(t-2v) & 12(t^2-2ut-2vt+2uv) & 12u(2t-u-2v) & 12u^2 & \\ 12t^2 & -12t(t-2u) & -12u(2t-u) & -12u^2t & 0 \end{array} \tag{30}$$

The partial derivatives $\partial^2 B/\partial v^2$ are

$$\begin{array}{lllll} 12v^2 & & & & \\ 24v(t-v) & 24uv & & & \\ 12(t^2-4vt+v^2) & 24u(t-2v) & 12u^2 & & \\ -24t(t-v) & -24u(2t-v) & -24u^2 & 0 & \\ 12t^2 & 24ut & 12u^2 & 0 & 0 \end{array} \tag{31}$$

The control points $\mathbf{P}_{i,j,k}$ are defined in terms of three constants $a_0 = (\sqrt{3}-1)/\sqrt{3}$, $a_1 = (\sqrt{3}+1)/(2\sqrt{3})$, and $a_2 = 1-(5-\sqrt{2})(7-\sqrt{3})/46$,

$$\begin{array}{lllll} \mathbf{P}_{040} & & & & \\ \mathbf{P}_{031} & \mathbf{P}_{130} & & & \\ \mathbf{P}_{022} & \mathbf{P}_{121} & \mathbf{P}_{220} & & \\ \mathbf{P}_{013} & \mathbf{P}_{112} & \mathbf{P}_{211} & \mathbf{P}_{310} & \\ \mathbf{P}_{004} & \mathbf{P}_{103} & \mathbf{P}_{202} & \mathbf{P}_{301} & \mathbf{P}_{400} \end{array} = \begin{array}{lllll} (0,1,0) & & & & \\ (0,1,a_0) & (a_0,1,0) & & & \\ (0,a_1,a_1) & (a_2,1,a_2) & (a_1,a_1,0) & & \\ (0,a_0,1) & (a_2,a_2,1) & (1,a_2,a_2) & (1,a_0,0) & \\ (0,0,1) & (a_0,0,1) & (a_1,0,a_1) & (1,0,a_0) & (1,0,0) \end{array} \tag{32}$$

The control weights $w_{i,j,k}$ are defined in terms of four constants, $b_0 = 4\sqrt{3}(\sqrt{3}-1)$, $b_1 = 3\sqrt{2}$, $b_2 = 4$, and $b_3 = \sqrt{2}(3+2\sqrt{2}-\sqrt{3})/\sqrt{3}$,

$$\begin{array}{lllll} w_{040} & & & & \\ w_{031} & w_{130} & & & \\ w_{022} & w_{121} & w_{220} & & \\ w_{013} & w_{112} & w_{211} & w_{310} & \\ w_{004} & w_{103} & w_{202} & w_{301} & w_{400} \end{array} = \begin{array}{lllll} b_0 & & & & \\ b_1 & b_1 & & & \\ b_2 & b_3 & b_2 & & \\ b_1 & b_3 & b_3 & b_1 & \\ b_0 & b_1 & b_2 & b_1 & b_0 \end{array} \tag{33}$$

Both the numerator and denominator polynomial are quartic polynomials. Notice that each boundary curve of the triangle patch is a quartic polynomial of one variable that is exactly what was shown earlier for a quadrant of a circle.

3.2 A Hemisphere

The construction of a hemisphere as a NURBS surface of degree 3 in each of u and v is similar to that for the half circle. For the half circle, we had control points at the circular poles $(0,1)$ and $(0,-1)$, each with associated weight 1. We postulated two other control points, $(\alpha,\pm 1)$ and determined that $\alpha = 2$ and that the weight $w = 1/3$.

The idea extends to 3D. We will select two control points at the circular poles $(0,0,1)$ and $(0,0,-1)$. We can add control points of the form $(2,0,\pm 1)$ to form a hemicircle; each point has an associated weight $1/3$. Now we can add more control points of the form $(2,\beta,\pm 1)$ to extrude the hemicircle to a hemisphere; each point has an associated weight w. To obtain a tensor product surface, we need a rectangular array of control points. The poles account for 2 and the other points account for 8. To pinch off the surface at the poles, we can require each pole to occur 4 times. We then have 16 control points to work with. Symbolic manipulation to force $x(u)^2 + y(u)^2 + z(u)^2 = 1$ leads to $\beta^2 = 16$ and $w = 1/9$. Choose the hemisphere where $y \geq 0$, so $\beta = 4$.

The knot vectors are the standard uniform ones, $u_i = v_i = 0$ for $0 \leq i \leq 3$ and $u_i = v_i = 1$ for $4 \leq i \leq 7$. Table 2 lists the control points, weights, and basis functions.

Table 2. The control points, weights, and basis functions for the half sphere. Each cell has the control point $\mathbf{C}_{ij}$ and the weight w_{ij}. The basis function corresponding to that term is formed from the polynomials that tag the row and column.

	$(1-u)^3$	$3(1-u)^2u$	$3(1-u)u^2$	u^3
$(1-v)^3$	$(0,0,1),\quad 1$	$(0,0,1),\quad 1/3$	$(0,0,1),\quad 1/3$	$(0,0,1),\quad 1$
$3(1-v)^2v$	$(2,0,1),\quad 1/3$	$(2,4,1),\quad 1/9$	$(-2,4,1),\quad 1/9$	$(-2,0,1),\quad 1/3$
$3(1-v)v^2$	$(2,0,-1),\quad 1/3$	$(2,4,-1),\quad 1/9$	$(-2,4,-1),\quad 1/9$	$(-2,0,-1),\quad 1/3$
v^3	$(0,0,-1),\quad 1$	$(0,0,-1),\quad 1/3$	$(0,0,-1),\quad 1/3$	$(0,0,-1),\quad 1$

(34)

Mathematica was used to verify symbolically that $x^2+y^2+z^2=1$.

3.3 Full Sphere

A full sphere can be formed from two hemispheres by splicing in a manner similar to that for generating a circle from two half circles. The u-knot vector is the standard uniform one with $u_i = 0$ for $0 \le i \le 3$ and $u_i = 1$ for $4 \le 7$. However, the v-knot vector has 11 elements with $v_i = 0$ for $0 \le i \le 3$, $v_i = 1/2$ for $4 \le i \le 6$, and $v_i = 1$ for $7 \le i \le 10$. Tables 3 and 4 show the control points, weights, and B-spline functions.

Table 3. The control points, weights, and basis functions for the sphere when $v \in [0, 1/2)$. The basis function corresponding to that term is formed from the polynomials that tag the row and column.

	$(1-2v)^3$	$3(2v)(1-2v)^2$	$3(2v)^2(1-2v)$	$(2v)^3$	0	0	0
$(1-u)^3$	$(0,0,1),1$	$(0,0,1),1/3$	$(0,0,1),1/3$	$(0,0,1),1$	$(0,0,1),1/3$	$(0,0,1),1/3$	$(0,0,1),1$
$3(1-u)^2u$	$(2,0,1),1/3,$	$(2,4,1),1/9$	$(-2,4,1),1/9$	$(-2,0,1),1/3$	$(-2,-4,1),1/9$	$(2,-4,1),1/9$	$(2,0,1),1/3$
$3(1-u)u^2$	$(2,0,-1),1/3,$	$(2,4,-1),1/9$	$(-2,4,-1),1/9$	$(-2,0,-1),1/3$	$(-2,-4,-1),1/9$	$(2,-4,-1),1/9$	$(2,0,-1),1/3$
u^3	$(0,0,-1),1$	$(0,0,-1),1/3$	$(0,0,-1),1/3$	$(0,0,-1),1$	$(0,0,-1),1/3$	$(0,0,-1),1/3$	$(0,0,-1),1$

Table 4. The control points, weights, and basis functions for the sphere when $v \in [1/2, 1)$. The basis function corresponding to that term is formed from the polynomials that tag the row and column.

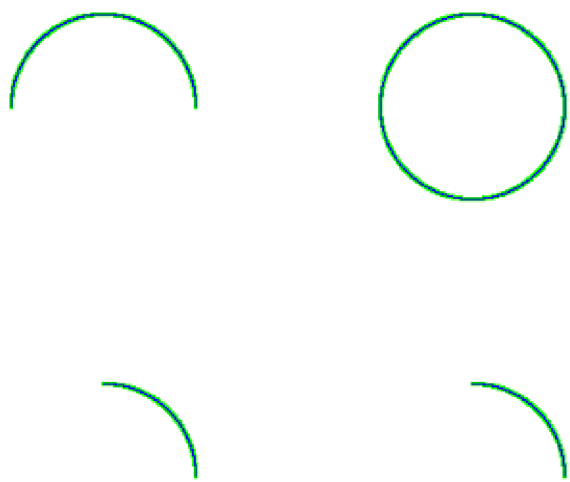
	0	0	0	$(2-2v)^3$	$3(2v-1)(2-2v)^2$	$3(2v-1)^2(2-2v)$	$(2v-1)^3$
$(1-u)^3$	$(0,0,1),1$	$(0,0,1),1/3$	$(0,0,1),1/3$	$(0,0,1),1$	$(0,0,1),1/3$	$(0,0,1),1/3$	$(0,0,1),1$
$3(1-u)^2u$	$(2,0,1),1/3,$	$(2,4,1),1/9$	$(-2,4,1),1/9$	$(-2,0,1),1/3$	$(-2,-4,1),1/9$	$(2,-4,1),1/9$	$(2,0,1),1/3$
$3(1-u)u^2$	$(2,0,-1),1/3,$	$(2,4,-1),1/9$	$(-2,4,-1),1/9$	$(-2,0,-1),1/3$	$(-2,-4,-1),1/9$	$(2,-4,-1),1/9$	$(2,0,-1),1/3$
u^3	$(0,0,-1),1$	$(0,0,-1),1/3$	$(0,0,-1),1/3$	$(0,0,-1),1$	$(0,0,-1),1/3$	$(0,0,-1),1/3$	$(0,0,-1),1$

The control points $(0, 0, 1)$ and $(0, 0, -1)$ each occur 7 times to pinch off the surface at the poles. The control points $(2, 0, 1)$ and $(2, 0, -1)$ each occur 2 times to wrap the surface around the up-axis.

4 Implementation

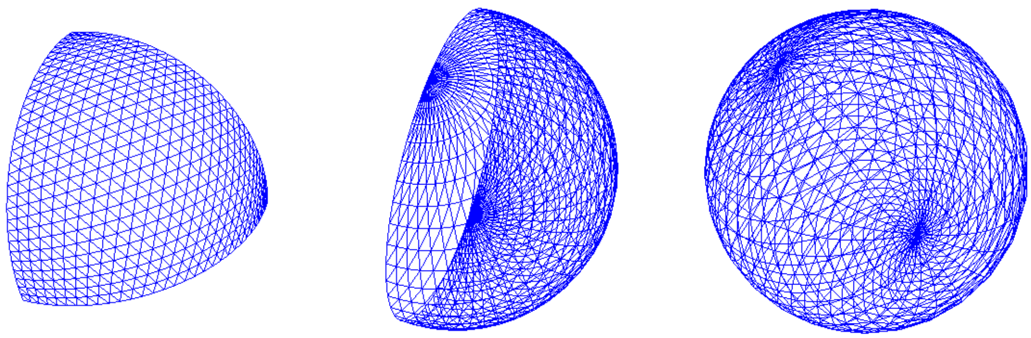
The implementations for NURBS circles are found in NURBSCircle.h and a sample application illustrating this is in the folder GeometricTools/GTE/Samples/Mathematics/NURBSCircle. Figure 1 shows a screen capture with 4 circular components.

Figure 1. A screen capture from the sample application NURBSCircle. The thick green curves are circular components computed using trigonometric functions. The thin blue curves are the same components computed using NURBS. The lower left is a quarter circle of degree 2 (Section 2.1.1), the lower right is a quarter circle of degree 4 (Section 2.1.2), the upper left is a half circle of degree 3 (Section 2.2) and the upper right is a full circle of degree 3 (Section 2.3).



The implementations for NURBS spheres are found in NURBSSphere.h and a sample application illustrating this is in the folder GeometricTools/GTE/Samples/Mathematics/NURBSSphere. Figure 2 shows screen captures with 3 spherical components.

Figure 2. Screen captures from the sample application NURBSSphere. The left image has an eighth sphere of degree 4 (Section 3.1.2). This is a triangle-patch surface. The middle image has a half sphere of degree 3 in each of its 2 parameters (Section 3.2). The right image has a full sphere of degree 3 in each of its 2 parameters (Section 3.3). The half and full spheres are rectangular tensor-product patches.



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