

# Representing a Circle or a Sphere with NURBS

David Eberly, Geometric Tools, Redmond WA 98052

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This document describes representing circles and spheres with NURBS. For more information about NURBS, a good engineering-style approach to NURBS is [3]. An excellent book covering the mathematics and modeling with splines is [1]. A more mathematically advanced presentation is [2].

## 1 Introduction

A NURBS curve in 2D is generated by a homogeneous B-spline curve in 3D,

$$(\bar{x}(u), \bar{y}(u), w(u)) = \sum_{i=0}^n N_{i,d}(u) w_i(\mathbf{C}_i, 1) \quad (1)$$

where the 3D curve has  $n + 1$  control points  $(\mathbf{C}_i, 1)$  [the  $\mathbf{C}_i$  are 2-tuples],  $n + 1$  weights  $w_i$ , and degree  $d$  with  $1 \leq d \leq n$ . The functions  $N_{i,d}(u)$  are the B-spline basis functions, which are defined recursively and require selection of a sequence of nondecreasing scalars  $u_i$  for  $0 \leq i \leq n + d + 1$ . Each  $u_i$  is called a knot and the total sequence is called a knot vector. The basis function that starts the recursive definition is

$$N_{i,0}(u) = \begin{cases} 1, & u_i \leq u < u_{i+1} \\ 0, & \text{otherwise} \end{cases} \quad (2)$$

for  $0 \leq i \leq n + d$ . The recursion is

$$N_{i,j}(u) = \frac{u - u_i}{u_{i+j} - u_i} N_{i,j-1}(u) + \frac{u_{i+j+1} - u}{u_{i+j+1} - u_{i+1}} N_{i+1,j-1}(u) \quad (3)$$

for  $1 \leq j \leq d$  and  $0 \leq i \leq n + d - j$ . Equation (1) uses only the final evaluations in the recursion,  $N_{i,d}(u)$  for which  $j = d$  and  $0 \leq i \leq n$ . The 2D curve is obtained by the perspective division,

$$(x(u), y(u)) = (\bar{x}(u), \bar{y}(u))/w(u) \quad (4)$$

In the section on circle representations, I will discuss NURBS curves that produce circular arcs; each curve will be listed with its  $n$ ,  $d$ ,  $u_i$ ,  $\mathbf{C}_i$ , and  $w_i$ .

A tensor-product NURBS surface in 3D is generated by a homogeneous B-spline surface in 4D,

$$(\bar{x}(u, v), \bar{y}(u, v), \bar{z}(u, v), w(u, v)) = \sum_{i_0=0}^{n_0} \sum_{i_1=0}^{n_1} N_{i_0,d_0}(u) N_{i_1,d_1}(v) w_{i_0 i_1}(\mathbf{C}_{i_0 i_1}, 1) \quad (5)$$

where the 4D surface has an  $(n_0 + 1) \times (n_1 + 1)$  array of control points  $(\mathbf{C}_{i_0 i_1}, 1)$  [the  $\mathbf{C}_{i_0 i_1}$  are 3-tuples], an  $(n_0 + 1) \times (n_1 + 1)$  array of weights  $w_{i_0 i_1}$ , and degrees  $d_0$  and  $d_1$  with  $1 \leq d_0 \leq n_0$  and  $1 \leq d_1 \leq n_1$ . The functions  $N_{i_0,d_0}(u)$  and  $N_{i_1,d_1}(v)$  each have a knot vector and are B-spline basis functions generated by equations (2) and (3). The 3D surface is obtained by the perspective division,

$$(x(u, v), y(u, v), z(u, v)) = (\bar{x}(u, v), \bar{y}(u, v), \bar{z}(u, v))/w(u, v) \quad (6)$$

In the section on sphere representations, I will discuss NURBS surfaces that produce spherical regions; each surface will be listed with its  $n_0$ ,  $n_1$ ,  $d_0$ ,  $d_1$ ,  $u_i$ ,  $v_i$ ,  $\mathbf{C}_{i_0 i_1}$ , and  $w_{i_0 i_1}$ .

## 2 Representing a Circle

Several representations are provided for a quarter of a circle in the first quadrant. A representation is also provided for half a circle. All these representation use the same knot pattern involving only the repeated zeros and ones at the beginning and end of the knot array. Finally, a representation for a full circle is created from that of the half circle by introducing a repeated knot of one-half that has the effect of splicing together two regular NURBS surfaces.

### 2.1 Quarter Circle

Consider the quarter circle in the first quadrant,  $x^2 + y^2 = 1$  with  $x \geq 0$  and  $y \geq 0$ .

#### 2.1.1 Degree 2

A quadrant of a circle can be represented as a NURBS curve of degree 2. The curve is  $x^2 + y^2 = 1$ ,  $x \geq 0$ ,  $y \geq 0$ . The general parameterization is

$$(\bar{x}(u), \bar{y}(u), w(u)) = w_0(1-u)^2(1, 0, 1) + w_1 2u(1-u)(1, 1, 1) + w_2 u^2(0, 1, 1) \quad (7)$$

for  $u \in [0, 1]$ . The parameterization is a ratio of quadratic polynomials,  $(x(u), y(u)) = (\bar{x}(u)/w(u), \bar{y}(u)/w(u))$ . The requirement that  $x(u)^2 + y(u)^2 = 1$  leads to the constraint  $2w_1^2 = w_0 w_2$ .

The choice of weights  $w_0 = 1$ ,  $w_1 = 1$ , and  $w_2 = 2$  leads to the parameterization

$$(x(u), y(u)) = \frac{(1-u^2, 2u)}{1+u^2} \quad (8)$$

If you were to tessellate the curve with an odd number of uniform samples of  $u$ , say  $u_i = i/(2n)$  for  $0 \leq i \leq 2n$ , then the resulting polyline is not symmetric about the midpoint  $u = 1/2$ .

To obtain a symmetric tessellation you need to choose  $w_0 = w_2$ . The weight constraint then implies  $w_0 = w_1 \sqrt{2}$ . The parameterization is then

$$(x(u), y(u)) = \frac{(\sqrt{2}(1-u)^2 + 2u(1-u), 2u(1-u) + \sqrt{2}u^2)}{\sqrt{2}(1-u)^2 + 2u(1-u) + \sqrt{2}u^2} \quad (9)$$

The NURBS parameters are the following. There are 3 control points, so  $n = 2$ . The degree is  $d = 2$ . The knot vector is  $\{u_0, u_1, u_2, u_3, u_4, u_5\} = \{0, 0, 0, 1, 1, 1\}$ . The control values are  $\mathbf{C}_0 = (1, 0)$ ,  $\mathbf{C}_1 = (1, 1)$ , and  $\mathbf{C}_2 = (0, 1)$ . The weights are  $w_0$ ,  $w_1$ , and  $w_2$  subject to the constraint  $2w_1^2 = w_0 w_2$ .

#### 2.1.2 Degree 4

An algebraic construction of the same type produces a ratio of quartic polynomials. The control points and control weights are required to be symmetric to obtain a tessellation that is symmetric about its midpoint. Define  $P_{d,k}(u) = (1-u)^{d-k}u^k$  for  $d > 0$  and  $0 \leq k \leq d$ . The NURBS curve is

$$\begin{aligned} (\bar{x}(u), \bar{y}(u), w(u)) = & P_{4,0}(u)w_0(1, 0, 1) + 4P_{4,1}(u)w_1(x_1, y_1, 1) + 6P_{4,2}(u)w_2(x_2, x_2, 1) \\ & + 4P_{4,3}(u)w_1(y_1, x_1, 1) + P_{4,4}(u)w_0(0, 1, 1) \end{aligned} \quad (10)$$

The parameterization is a ratio of quartic polynomials,  $(x(u), y(u)) = (\bar{x}(u)/w(u), \bar{y}(u)/w(u))$ . We have one degree of freedom, choosing  $w_0 = 1$  which effectively divides the numerators and denominators by  $w_0$ .

To be a circle, we need  $x(u)^2 + y(u)^2 = 1$ , which is equivalent to  $\bar{x}^2(u) + \bar{y}^2(u) - w^2(u) = 0$ . Some algebraic manipulation leads to

$$\begin{aligned}
0 = & [P_{8,1}(u) + P_{8,7}(u)][8w_1(x_1 - 1)] + \\
& [P_{8,2}(u) + P_{8,6}(u)][12w_2(x_2 - 1) + 16w_1^2(x_1^2 + y_1^2 - 1)] + \\
& [P_{8,3}(u) + P_{8,5}(u)][8w_1(y_1 - 1) + 48w_1w_2(x_2(x_1 + y_1) - 1)] + \\
& [P_{8,4}(u) + P_{8,4}(u)][16w_1^2(2x_1y_1 - 1) + 18w_2^2(2x_2^2 - 1) - 1]
\end{aligned} \tag{11}$$

The four  $u$ -polynomials are linearly independent in the vector space of  $u$ -polynomials, which means their four coefficients must all be zero,

$$\begin{aligned}
w_1(x_1 - 1) &= 0 \\
3w_2(x_2 - 1) + 4w_1^2(x_1^2 + y_1^2 - 1) &= 0 \\
w_1(y_1 - 1) + 6w_1w_2(x_2(x_1 + y_1) - 1) &= 0 \\
16w_1^2(2x_1y_1 - 1) + 18w_2^2(2x_2^2 - 1) - 1 &= 0
\end{aligned} \tag{12}$$

The solution space is constrained to  $w_1 \geq 0$ ,  $w_2 \geq 0$ ,  $x_1 \geq 0$ ,  $y_1 \geq 0$  and  $x_2 \geq 0$ . It cannot be that  $w_2 = 0$ ; for if it were, the fourth equation requires  $w_1 \neq 0$ . The first equation implies  $x_1 = 1$  and the third equation implies  $y_1 = 1$ . The second equation then implies  $w_1 = 0$ , which is a contradiction. The conclusion is that  $w_2 > 0$ .

If  $w_1 = 0$ , then  $x_2 = 1$  and  $w_2 = 1/\sqrt{18}$ . The values of  $x_1$  and  $y_1$  are irrelevant, because  $w_1 = 0$  annihilates the  $(x_1, y_1, 1)$  and  $(y_1, x_1, 1)$  terms in equation (10), leading to the parameterization

$$(\bar{x}(u), \bar{y}(u), w(u)) = (1 - u)^4(1, 0, 1) + \sqrt{2}(1 - u)^2u^2(1, 1, 1) + u^4(0, 1, 1) \tag{13}$$

A plot of the parameterized curve  $(\bar{x}(u)/w(u), \bar{y}(u)/w(u))$  shows that the curve is indeed a quarter circle. Multiplying numerators and denominator of the parameterized curve by  $\sqrt{2}$  leads to a parameterization similar to that of equation (9).

If  $w_1 \neq 0$ , then  $x_1 = 1$ . Using Mathematica [4], the equations (12) were solved with free variable  $w_2$  to produce

$$w_1 = \frac{3w_2}{2\sqrt{2}}, \quad x_2 = 1 - \frac{1}{6w_2}, \quad y_1 = \frac{1}{3w_2} \tag{14}$$

or

$$w_1 = \frac{1}{2} \sqrt{\frac{1}{2} + \left(1 + \frac{1}{\sqrt{2}}\right)(3w_2 - 1)}, \quad x_2 = \frac{(3w_2 + 1)\sqrt{2} - 1}{6w_2}, \quad y_1 = \sqrt{2} - 1 \tag{15}$$

where  $w_2 \geq 1/3$ .

Thus, we have at least one more degree of freedom and there are infinitely many solutions. For example, if we choose  $w_1 = 1$ , then there are 2 possibilities for the remaining parameters:  $(y_1, x_2, w_2) = (1/(2\sqrt{2}), 1 - \sqrt{2}/8, 2\sqrt{2}/3)$  and  $(y_1, x_2, w_2) = (\sqrt{2} - 1, (44\sqrt{2} - 1)/79, (16 - 7\sqrt{2})/6)$ . With these choices, Mathematica verified that  $x(u)^2 + y(u)^2 = 1$  and parametric plots of  $(x(u), y(u))$  were indeed quarter circles. Mathematica

also symbolically generated solutions to the constraints when asked to solve for  $y_1$ ,  $x_2$ , and  $w_2$  in terms of  $w_1$ .

The NURBS parameters are the following. There are 5 control points, so  $n = 4$ . The degree is  $d = 4$ . The knot vector has 10 components with  $u_i = 0$  for  $0 \leq i \leq 4$  and  $u_i = 1$  for  $5 \leq i \leq 9$ . Assuming the constraints of equation (12) are satisfied, the control values are  $\mathbf{C}_0 = (1, 0)$ ,  $\mathbf{C}_1 = (x_1, y_1)$ ,  $\mathbf{C}_2 = (x_2, x_2)$ ,  $\mathbf{C}_3 = (y_1, x_1)$ , and  $\mathbf{C}_4 = (0, 1)$  and the weights are  $w_0 = w_4 = 1$ ,  $w_2$ , and  $w_1 = w_3$ .

## 2.2 Half Circle

This section describes a symmetric parameterization of the half circle  $x^2 + y^2 = 1$  for  $y \geq 0$ . The NURBS curve has degree 3. The control values are chosen to be  $(1, 0)$ ,  $(1, \alpha)$ ,  $(-1, \alpha)$  and  $(-1, 0)$  where  $\alpha > 0$  and the weights are chosen to be  $w_0 = w_3 = 1$  and  $w_1 = w_2 = \omega > 0$ . The parameters  $\alpha$  and  $\omega$  must be chosen so that we indeed obtain a half circle. The curve is

$$(x(u), y(u)) = \frac{(1-u)^3(1,0) + 3u(1-u)^2\omega(1,\alpha) + 3u^2(1-u)\omega(-1,\alpha) + u^3(-1,0)}{(1-u)^3 + 3u(1-u)^2\omega + 3u^2(1-u)\omega + u^3} \quad (16)$$

Symbolic manipulation of  $x(u)^2 + y(u)^2 = 1$  leads to the constraints  $\omega = 1/3$  and  $\alpha = 2$ .

The NURBS parameters are the following. There are 4 control points, so  $n = 3$ . The degree is  $d = 3$ . The knot vector has 8 components with  $u_i = 0$  for  $0 \leq i \leq 3$  and  $u_i = 1$  for  $4 \leq i \leq 7$ . The control values are  $\mathbf{C}_0 = (1, 0)$ ,  $\mathbf{C}_1 = (1, 2)$ ,  $\mathbf{C}_2 = (-1, 2)$ , and  $\mathbf{C}_3 = (-1, 0)$ . The weights are  $w_0 = w_3 = 1$  and  $w_1 = w_2 = 1/3$ .

## 2.3 Full Circle

The full-circle curve can be obtained by splicing together two half-circle curves as constructed in the previous section ( $d = 3$ ). The 7 control values ( $n = 6$ ) are in the counterclockwise order  $(1, 0)$ ,  $(1, 2)$ ,  $(-1, 2)$ ,  $(-1, 0)$ ,  $(-1, -2)$ ,  $(1, -2)$ , and  $(1, 0)$ . Observe that the first and last controls are duplicated in order to close the curve. The splicing occurs at the control value  $(0, -1)$ . Rather than duplicate the control, the splicing is accomplished by choosing an interior knot of  $1/2$  with multiplicity 3; the knots are  $u_i = 0$  for  $0 \leq i \leq 3$ ,  $u_i = 1/2$  for  $4 \leq i \leq 6$ , and  $u_i = 1$  for  $7 \leq i \leq 10$ . Note that the number of knots is as required:  $n+d+2 = 11$ .

Equation (2) provides the initial functions for starting the recursion. These are  $N_{i,0}(u) = 0$  for  $i \in \{0, 1, 2, 4, 5, 7, 8, 9\}$  and

$$\begin{aligned} N_{3,0}(u) &= \{1, \quad u \in [0, 1/2); \quad 0, \quad u \in [1/2, 1)\} \\ N_{6,0}(u) &= \{0, \quad u \in [0, 1/2); \quad 1, \quad u \in [1/2, 1)\} \end{aligned} \quad (17)$$

For  $j = 1$ , equation (3) produces  $N_{i,1}(u) = 0$  for  $i \in \{0, 1, 4, 7, 8\}$  and

$$\begin{aligned} N_{2,1}(u) &= \{1 - 2u, \quad u \in [0, 1/2); \quad 0, \quad u \in [1/2, 1)\} \\ N_{3,1}(u) &= \{2u, \quad u \in [0, 1/2); \quad 0, \quad u \in [1/2, 1)\} \\ N_{5,1}(u) &= \{0, \quad u \in [0, 1/2); \quad 2 - 2u, \quad u \in [1/2, 1)\} \\ N_{6,1}(u) &= \{0, \quad u \in [0, 1/2); \quad 2u - 1, \quad u \in [1/2, 1)\} \end{aligned} \quad (18)$$

For  $j = 2$ , equation (3) produces  $N_{2,2}(u) = 0$ ,  $N_{7,2}(u) = 0$ , and

$$\begin{aligned}
N_{1,2}(u) &= \{(1 - 2u)^2, & u \in [0, 1/2); & 0, & u \in [1/2, 1)\} \\
N_{2,2}(u) &= \{2(2u)(1 - 2u), & u \in [0, 1/2); & 0, & u \in [1/2, 1)\} \\
N_{3,2}(u) &= \{(2u)^2, & u \in [0, 1/2); & 0, & u \in [1/2, 1)\} \\
N_{4,2}(u) &= \{0, & u \in [0, 1/2); & (2 - 2u)^2, & u \in [1/2, 1)\} \\
N_{5,2}(u) &= \{0, & u \in [0, 1/2); & 2(2u - 1)(2 - 2u), & u \in [1/2, 1)\} \\
N_{6,2}(u) &= \{0, & u \in [0, 1/2); & (2u - 1)^2, & u \in [1/2, 1)\}
\end{aligned} \tag{19}$$

For  $j = 3$ , equation (3) produces

$$\begin{aligned}
N_{0,3}(u) &= \{(1 - 2u)^3, & u \in [0, 1/2); & 0, & u \in [1/2, 1)\} \\
N_{1,3}(u) &= \{3(2u)(1 - 2u)^2, & u \in [0, 1/2); & 0, & u \in [1/2, 1)\} \\
N_{2,3}(u) &= \{3(2u)^2(1 - 2u), & u \in [0, 1/2); & 0, & u \in [1/2, 1)\} \\
N_{3,3}(u) &= \{(2u)^3, & u \in [0, 1/2); & (2 - 2u)^3, & u \in [1/2, 1)\} \\
N_{4,3}(u) &= \{0, & u \in [0, 1/2); & 3(2u - 1)(2 - 2u)^2, & u \in [1/2, 1)\} \\
N_{5,3}(u) &= \{0, & u \in [0, 1/2); & 3(2u - 1)^2(2 - 2u), & u \in [1/2, 1)\} \\
N_{6,3}(u) &= \{0, & u \in [0, 1/2); & (2u - 1)^3, & u \in [1/2, 1)\}
\end{aligned} \tag{20}$$

The last set of functions,  $N_{i,3}(u)$ , are what are used in the NURBS curve definition. Observe that the only function that has nonzero expressions for both  $u \in [0, 1/2)$  and  $u \in [1/2, 1)$  is  $N_{3,3}(u)$ . This is where the splicing occurs. The weights are the same as in the half-circle case. Table 1 summarizes the control points and weights.

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**Table 1.** The control points, weights, and intervals of the domain for the circle.

controls	(1, 0)	(1, 2)	(-1, 2)	(-1, 0)	(-1, -2)	(1, -2)	(1, 0)
weights	1	1/3	1/3	1	1/3	1/3	1
$u \in [0, 1/2)$	$(1 - 2u)^3$	$3(2u)(1 - 2u)^2$	$3(2u)^2(1 - 2u)$	$(2u)^3$	0	0	0
$u \in [1/2, 1)$	0	0	0	$(2 - 2u)^3$	$3(2u - 1)(2 - 2u)^2$	$3(2u - 1)^2(2 - 2u)$	$(2u - 1)^3$

---

For  $u \in [0, 1/2)$ , the NURBS curve has nonzero weights only for the first four control points. For  $u \in [1/2, 1)$ , the NURBS curve has nonzero weights only for the last four control points.

## 3 Representing a Sphere

### 3.1 One Octant of a Sphere

#### 3.1.1 Nonsymmetric, Degree 4

An octant of a sphere can be represented as a triangular NURBS surface patch of degree 4. A simple parameterization of  $x^2 + y^2 + z^2 = 1$  can be made by setting  $r^2 = x^2 + y^2$ . The sphere is then  $r^2 + z^2 = 1$ . Now apply the parameterization for a circle,

$$(r, z) = \frac{(1 - u^2, 2u)}{1 + u^2} \quad (21)$$

But  $(x/r)^2 + (y/r)^2 = 1$ , so another application of the parameterization for a circle is

$$\frac{(x, y)}{r} = \frac{(1 - v^2, 2v)}{1 + v^2} \quad (22)$$

Combining these produces

$$(x(u, v), y(u, v), z(u, v)) = \frac{((1 - u^2)(1 - v^2), (1 - u^2)2v, 2u(1 + v^2))}{(1 + u^2)(1 + v^2)} \quad (23)$$

The components are ratios of quartic polynomials. The domain is  $u \geq 0, v \geq 0$ , and  $u + v \leq 1$ . In barycentric coordinates, set  $w = 1 - u - v$  so that  $u + v + w = 1$  with  $u, v$ , and  $w$  nonnegative. In this setting, you can think of the octant of the sphere as a mapping from the  $uvw$ -triangle with vertices  $(1, 0, 0)$ ,  $(0, 1, 0)$ , and  $(0, 0, 1)$ . Although a valid parameterization, a symmetric subdivision of the the  $uvw$ -triangle does not lead to a symmetric tessellation of the sphere.

#### 3.1.2 Symmetric, Degree 4

Another parameterization is provided in [2]. This one chooses symmetric control points and symmetric weights,

$$(x(u, v), y(u, v), z(u, v)) = \frac{\sum_{i=0}^4 \sum_{j=0}^{4-i} w_{i,j,4-i-j} \mathbf{P}_{i,j,4-i-j} B_{i,j}(u, v)}{\sum_{i=0}^4 \sum_{j=0}^{4-i} w_{i,j,4-i-j} B_{i,j}(u, v)} \quad (24)$$

where

$$B_{i,j}(u, v) = \frac{4!}{i!j!(4-i-j)!} u^i v^j t^{4-i-j}, \quad u \geq 0, \quad v \geq 0, \quad t = 1 - u - v \geq 0 \quad (25)$$

are the Bernstein polynomials. These can be written in a triangular form,

$$\begin{array}{cccccc} v^4 & & & & & \\ 4v^3t & 4uv^3 & & & & \\ 6v^2t^2 & 12wv^2t & 6u^2v^2 & & & \\ 4vt^3 & 12wvt^2 & 12u^2vt & 4u^3v & & \\ t^4 & 4ut^3 & 6u^2t^2 & 4u^3t & u^4 & \end{array} \quad (26)$$

In the implementation I allow for derivative computation through order 2. The partial derivatives  $\partial B/\partial u$  are

$$\begin{array}{cccccc}
0 & & & & & \\
-4v^3 & 4v^3 & & & & \\
-12v^2t & 12v^2(t-u) & 12uv^2 & & & \\
-12vt^2 & 12vt(t-2u) & 12uv(2t-u) & 12u^2v & & \\
-4t^3 & 4t^2(t-3u) & 12ut(t-u) & 4u^2(3t-u) & 4u^3 & 
\end{array} \tag{27}$$

The partial derivatives  $\partial B/\partial v$  are

$$\begin{array}{cccccc}
4v^3 & & & & & \\
4v^2(3t-v) & 12uv^2 & & & & \\
12vt(t-v) & 12uv(2t-v) & 12u^2v & & & \\
4t^2(t-3v) & 12ut(t-2v) & 12u^2(t-v) & 4u^3 & & \\
-t^3 & -12ut^2 & -12u^2t & -4u^3 & 0 & 
\end{array} \tag{28}$$

The partial derivatives  $\partial^2 B/\partial u^2$  are

$$\begin{array}{cccccc}
0 & & & & & \\
0 & 0 & & & & \\
12v^2 & -24v^2 & 12v^2 & & & \\
24vt & -24v(2t-u) & 24v(t-2u) & 24uv & & \\
12t^2 & -24t(t-u) & 12(t^2-4ut+u^2) & 24u(t-u) & 12u^2 & 
\end{array} \tag{29}$$

The partial derivatives  $\partial^2 B/\partial u\partial v$  are

$$\begin{array}{cccccc}
0 & & & & & \\
-12v^2 & 12v^2 & & & & \\
-12v(2t-v) & 12v(2t-2u-v) & 24uv & & & \\
-12t(t-2v) & 12(t^2-2ut-2vt+2uv) & 12u(2t-u-2v) & 12u^2 & & \\
12t^2 & -12t(t-2u) & -12u(2t-u) & -12u^2t & 0 & 
\end{array} \tag{30}$$

The partial derivatives  $\partial^2 B/\partial v^2$  are

$$\begin{array}{cccccc}
12v^2 & & & & & \\
24v(t-v) & 24uv & & & & \\
12(t^2-4vt+v^2) & 24u(t-2v) & 12u^2 & & & \\
-24t(t-v) & -24u(2t-v) & -24u^2 & 0 & & \\
12t^2 & 24ut & 12u^2 & 0 & 0 & 
\end{array} \tag{31}$$



The control points  $\mathbf{P}_{i,j,k}$  are defined in terms of three constants  $a_0 = (\sqrt{3} - 1)/\sqrt{3}$ ,  $a_1 = (\sqrt{3} + 1)/(2\sqrt{3})$ , and  $a_2 = 1 - (5 - \sqrt{2})(7 - \sqrt{3})/46$ ,

$$\begin{array}{llllll}
\mathbf{P}_{040} & & & & & (0, 1, 0) \\
\mathbf{P}_{031} & \mathbf{P}_{130} & & & & (0, 1, a_0) \quad (a_0, 1, 0) \\
\mathbf{P}_{022} & \mathbf{P}_{121} & \mathbf{P}_{220} & = & (0, a_1, a_1) & (a_2, 1, a_2) \quad (a_1, a_1, 0) \\
\mathbf{P}_{013} & \mathbf{P}_{112} & \mathbf{P}_{211} & \mathbf{P}_{310} & & (0, a_0, 1) \quad (a_2, a_2, 1) \quad (1, a_2, a_2) \quad (1, a_0, 0) \\
\mathbf{P}_{004} & \mathbf{P}_{103} & \mathbf{P}_{202} & \mathbf{P}_{301} & \mathbf{P}_{400} & (0, 0, 1) \quad (a_0, 0, 1) \quad (a_1, 0, a_1) \quad (1, 0, a_0) \quad (1, 0, 0)
\end{array} \tag{32}$$

The control weights  $w_{i,j,k}$  are defined in terms of four constants,  $b_0 = 4\sqrt{3}(\sqrt{3} - 1)$ ,  $b_1 = 3\sqrt{2}$ ,  $b_2 = 4$ , and  $b_3 = \sqrt{2}(3 + 2\sqrt{2} - \sqrt{3})/\sqrt{3}$ ,

$$\begin{array}{llllllll}
w_{040} & & & & & & & b_0 \\
w_{031} & w_{130} & & & & & & b_1 \quad b_1 \\
w_{022} & w_{121} & w_{220} & = & b_2 & b_3 & b_2 & \\
w_{013} & w_{112} & w_{211} & w_{310} & & & & b_1 \quad b_3 \quad b_3 \quad b_1 \\
w_{004} & w_{103} & w_{202} & w_{301} & w_{400} & & & b_0 \quad b_1 \quad b_2 \quad b_1 \quad b_0
\end{array} \tag{33}$$

Both the numerator and denominator polynomial are quartic polynomials. Notice that each boundary curve of the triangle patch is a quartic polynomial of one variable that is exactly what was shown earlier for a quadrant of a circle.

### 3.2 A Hemisphere

The construction of a hemisphere as a NURBS surface of degree 3 in each of  $u$  and  $v$  is similar to that for the half circle. For the half circle, we had control points at the circular poles  $(0, 1)$  and  $(0, -1)$ , each with associated weight 1. We postulated two other control points,  $(\alpha, \pm 1)$  and determined that  $\alpha = 2$  and that the weight  $w = 1/3$ .

The idea extends to 3D. We will select two control points at the circular poles  $(0, 0, 1)$  and  $(0, 0, -1)$ . We can add control points of the form  $(2, 0, \pm 1)$  to form a hemicircle; each point has an associated weight  $1/3$ . Now we can add more control points of the form  $(2, \beta, \pm 1)$  to extrude the hemicircle to a hemisphere; each point has an associated weight  $w$ . To obtain a tensor product surface, we need a rectangular array of control points. The poles account for 2 and the other points account for 8. To pinch off the surface at the poles, we can require each pole to occur 4 times. We then have 16 control points to work with. Symbolic manipulation to force  $x(u)^2 + y(u)^2 + z(u)^2 = 1$  leads to  $\beta^2 = 16$  and  $w = 1/9$ . Choose the hemisphere where  $y \geq 0$ , so  $\beta = 4$ .

The knot vectors are the standard uniform ones,  $u_i = v_i = 0$  for  $0 \leq i \leq 3$  and  $u_i = v_i = 1$  for  $4 \leq i \leq 7$ . Table 2 lists the control points, weights, and basis functions.

**Table 2.** The control points, weights, and basis functions for the half sphere. Each cell has the control point  $\mathbf{C}_{ij}$  and the weight  $w_{ij}$ . The basis function corresponding to that term is formed from the polynomials that tag the row and column.

	$(1-u)^3$	$3(1-u)^2u$	$3(1-u)u^2$	$u^3$
$(1-v)^3$	(0, 0, 1), 1	(0, 0, 1), 1/3	(0, 0, 1), 1/3	(0, 0, 1), 1
$3(1-v)^2v$	(2, 0, 1), 1/3	(2, 4, 1), 1/9	(-2, 4, 1), 1/9	(-2, 0, 1), 1/3
$3(1-v)v^2$	(2, 0, -1), 1/3	(2, 4, -1), 1/9	(-2, 4, -1), 1/9	(-2, 0, -1), 1/3
$v^3$	(0, 0, -1), 1	(0, 0, -1), 1/3	(0, 0, -1), 1/3	(0, 0, -1), 1

Mathematica was used to verify symbolically that  $x^2 + y^2 + z^2 = 1$ .

### 3.3 Full Sphere

A full sphere can be formed from two hemispheres by splicing in a manner similar to that for generating a circle from two half circles. The  $u$ -knot vector is the standard uniform one with  $u_i = 0$  for  $0 \leq i \leq 3$  and  $u_i = 1$  for  $4 \leq 7$ . However, the  $v$ -knot vector has 11 elements with  $v_i = 0$  for  $0 \leq i \leq 3$ ,  $v_i = 1/2$  for  $4 \leq i \leq 6$ , and  $v_i = 1$  for  $7 \leq i \leq 10$ . Tables 3 and 4 show the control points, weights, and B-spline functions.

**Table 3.** The control points, weights, and basis functions for the sphere when  $v \in [0, 1/2)$ . The basis function corresponding to that term is formed from the polynomials that tag the row and column.

	$(1-2v)^3$	$3(2v)(1-2v)^2$	$3(2v)^2(1-2v)$	$(2v)^3$	0	0	0
$(1-u)^3$	(0, 0, 1), 1	(0, 0, 1), 1/3	(0, 0, 1), 1/3	(0, 0, 1), 1	(0, 0, 1), 1/3	(0, 0, 1), 1/3	(0, 0, 1), 1
$3(1-u)^2u$	(2, 0, 1), 1/3	(2, 4, 1), 1/9	(-2, 4, 1), 1/9	(-2, 0, 1), 1/3	(-2, -4, 1), 1/9	(2, -4, 1), 1/9	(2, 0, 1), 1/3
$3(1-u)u^2$	(2, 0, -1), 1/3	(2, 4, -1), 1/9	(-2, 4, -1), 1/9	(-2, 0, -1), 1/3	(-2, -4, -1), 1/9	(2, -4, -1), 1/9	(2, 0, -1), 1/3
$u^3$	(0, 0, -1), 1	(0, 0, -1), 1/3	(0, 0, -1), 1/3	(0, 0, -1), 1	(0, 0, -1), 1/3	(0, 0, -1), 1/3	(0, 0, -1), 1

**Table 4.** The control points, weights, and basis functions for the sphere when  $v \in [1/2, 1)$ . The basis function corresponding to that term is formed from the polynomials that tag the row and column.

	0	0	0	$(2-2v)^3$	$3(2v-1)(2-2v)^2$	$3(2v-1)^2(2-2v)$	$(2v-1)^3$
$(1-u)^3$	(0, 0, 1), 1	(0, 0, 1), 1/3	(0, 0, 1), 1/3	(0, 0, 1), 1	(0, 0, 1), 1/3	(0, 0, 1), 1/3	(0, 0, 1), 1
$3(1-u)^2u$	(2, 0, 1), 1/3	(2, 4, 1), 1/9	(-2, 4, 1), 1/9	(-2, 0, 1), 1/3	(-2, -4, 1), 1/9	(2, -4, 1), 1/9	(2, 0, 1), 1/3
$3(1-u)u^2$	(2, 0, -1), 1/3	(2, 4, -1), 1/9	(-2, 4, -1), 1/9	(-2, 0, -1), 1/3	(-2, -4, -1), 1/9	(2, -4, -1), 1/9	(2, 0, -1), 1/3
$u^3$	(0, 0, -1), 1	(0, 0, -1), 1/3	(0, 0, -1), 1/3	(0, 0, -1), 1	(0, 0, -1), 1/3	(0, 0, -1), 1/3	(0, 0, -1), 1

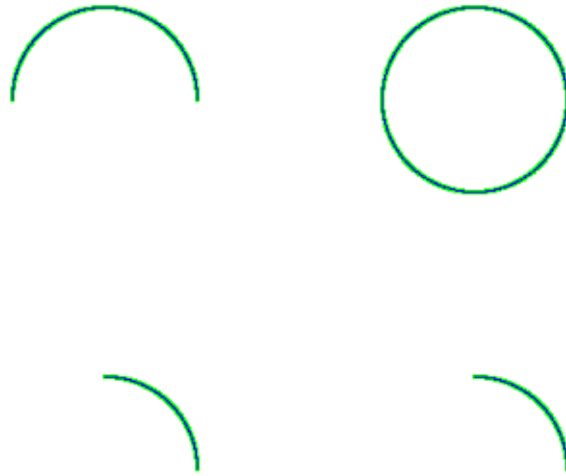
The control points (0, 0, 1) and (0, 0, -1) each occur 7 times to pinch off the surface at the poles. The control points (2, 0, 1) and (2, 0, -1) each occur 2 times to wrap the surface around the up-axis.

## 4 Implementation

The implementations for NURBS circles are found in [GteNURBSCircle.h](#) and a sample application illustrating this is in the folder `GeometricTools/GTEngine/Samples/Mathematics/NURBSCircle`. Figure 1 shows a screen capture with 4 circular components.

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**Figure 1.** A screen capture from the sample application `NURBSCircle`. The thick green curves are circular components computed using trigonometric functions. The thin blue curves are the same components computed using NURBS. The lower left is a quarter circle of degree 2 (Section 2.1.1), the lower right is a quarter circle of degree 4 (Section 2.1.2), the upper left is a half circle of degree 3 (Section 2.2) and the upper right is a full circle of degree 3 (Section 2.3).

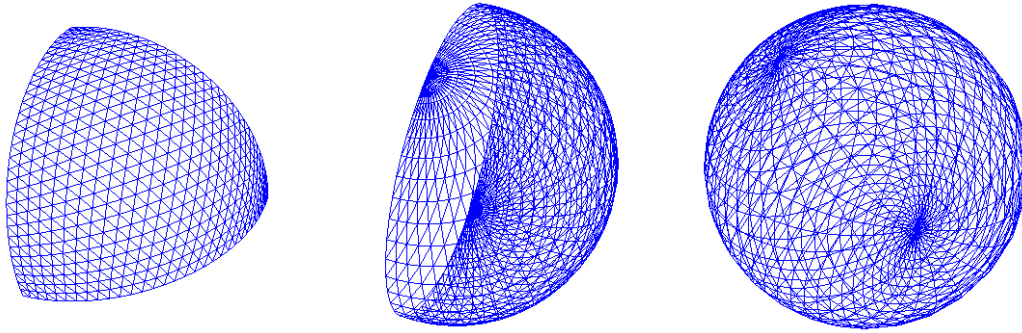


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The implementations for NURBS spheres are found in [GteNURBSSphere.h](#) and a sample application illustrating this is in the folder `GeometricTools/GTEngine/Samples/Mathematics/NURBSSphere`. Figure 2 shows screen captures with 3 spherical components.

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**Figure 2.** Screen captures from the sample application NURBSSphere. The left image has an eighth sphere of degree 4 (Section 3.1.2). This is a triangle-patch surface. The middle image has a half sphere of degree 3 in each of its 2 parameters (Section 3.2). The right image has a full sphere of degree 3 in each of its 2 parameters (Section 3.3). The half and full spheres are rectangular tensor-product patches.



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## References

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- [4] Wolfram Research, Inc. *Mathematica 11.3*. Wolfram Research, Inc., Champaign, Illinois, 2018.