Intersection of a Line and a Cone

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Contents

1 Introduction

A *infinite, single-sided, solid cone* has a vertex $\mathbf{V}$, an axis ray whose origin is $\mathbf{V}$ and unit-length direction is $\mathbf{D}$, and an acute cone angle $\theta \in (0, \pi/2)$. A point $\mathbf{X}$ is inside the cone when the angle between $\mathbf{D}$ and $\mathbf{X} - \mathbf{V}$ is in $[0, \theta]$. Algebraically, the containment is defined by

$$\mathbf{D} \cdot \frac{(\mathbf{X} - \mathbf{V})}{|\mathbf{X} - \mathbf{V}|} \geq \cos(\theta) \tag{1}$$

when $\mathbf{X} \neq \mathbf{V}$. Equivalently, the containment is defined by

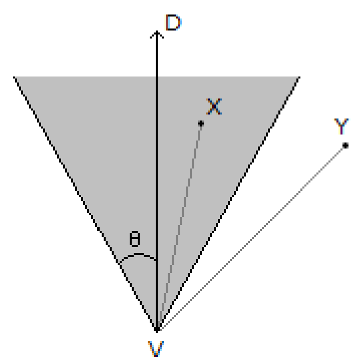
$$\mathbf{D} \cdot (\mathbf{X} - \mathbf{V}) \geq |\mathbf{X} - \mathbf{V}| \cos(\theta) \tag{2}$$

which includes the case $\mathbf{X} = \mathbf{V}$. Finally, we can avoid computing square roots in the implementation by squaring the dot-product equation to obtain a quadratic equation and requiring that only points above the supporting plane of the single-sided cone be considered. The definition is

$$Q(\mathbf{X}) = (\mathbf{D} \cdot (\mathbf{X} - \mathbf{V}))^2 - \cos^2(\theta)|\mathbf{X} - \mathbf{V}|^2 = (\mathbf{X} - \mathbf{V})^\mathsf{T} M (\mathbf{X} - \mathbf{V}) \geq 0, \;\; \mathbf{D} \cdot (\mathbf{X} - \mathbf{V}) \geq 0 \tag{3}$$

where $M = \mathbf{D}\mathbf{D}^\mathsf{T} - \cos^2(\theta) I$ is a symmetric 3×3 matrix with I the 3×3 identity matrix. Figure 1 shows a 2D cone, which is sufficient to illustrate the quantites in 3D.

Figure 1. A 2D view of a single-sided cone. $\mathbf{X}$ is inside the cone and $\mathbf{Y}$ is outside the cone.



Because of the constraint on θ, both $\cos(\theta) > 0$ and $\sin(\theta) > 0$. The single-sided cone is *finite* when you specify a maximum height h measured along the cone axis, in which case $\mathbf{D} \cdot (\mathbf{X} - \mathbf{V}) \leq h$. You may think of the infinite single-sided code having height $h = +\infty$.

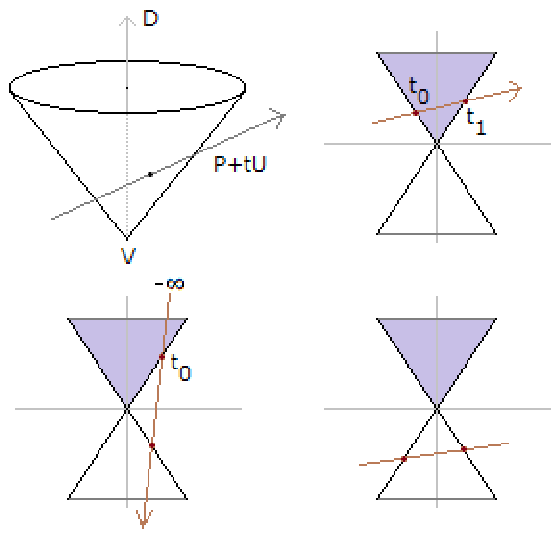
A line is parameterized by $\mathbf{X}(t) = \mathbf{C} + t\mathbf{U}$, where $\mathbf{C}$ is a point on the line, $\mathbf{U}$ is a unit-length direction vector for the line, and t is a real number. A ray has the subset of the line with restriction $t \geq 0$. A segment is a subset of the line with restriction $t \in [0, 1]$; however, a segment is typically defined using two endpoints, $\mathbf{E}_0$ and $\mathbf{E}_1$, with $\mathbf{X}(t) = (1 - t)\mathbf{E}_0 + t\mathbf{E}_1$.

2 Intersection with a Line

Let us find the points of intersection with the cone boundary $Q(\mathbf{X}) = 0$, where Q is defined by Equation (3). Substitute the line equation $\mathbf{X}(t) = \mathbf{P} + t\mathbf{U}$ to obtain $c_2^2 + 2c_1 t + c_0 = 0$, where $\mathbf{\Delta} = \mathbf{P} - \mathbf{V}$, $c_2 = \mathbf{U}^{\mathsf{T}} M \mathbf{U}$, $c_1 = \mathbf{U}^{\mathsf{T}} M \mathbf{\Delta}$, and $c_0 = \mathbf{\Delta}^{\mathsf{T}} M \mathbf{\Delta}$. The bounds $0 \leq \mathbf{D} \cdot (\mathbf{X}(t) - \mathbf{V}) \leq h$ become $0 \leq \mathbf{D} \cdot \mathbf{\Delta} + t\mathbf{D} \cdot \mathbf{U} \leq h$. We must compute the roots of a quadratic polynomial (possibly degenerate) subject to linear inequality constraints on t. The roots may be computed as if $h = +\infty$; the clamping to finite h can be applied as a postprocessing step.

Suppose that $c_2 \neq 0$. The formal roots are $t = (-c_1 \pm \sqrt{\delta})/c_2$, where $\delta = c_1^2 - c_0 c_2$. If $\delta < 0$, the quadratic polynomial has no real-valued roots, in which case the line does not intersect the double-sided cone. If $\delta = 0$, the polynomial has a repeated real-value root $t = -c_1/c_2$, in which case the line is tangent to the double-sided cone at a single point. If $\delta > 0$, the polynomial has two distinct real-valued roots, in which case the line penetrates the double-sided cone at two points. Figure 2 illustrates the various cases.

Figure 2. Geometric configurations when $c_2 \neq 0$. We care only about the intersections with the positive cone; that is, where $\mathbf{D} \cdot (\mathbf{X} - \mathbf{V}) \geq 0$. Upper left: $\delta = 0$, the line is tangent to the cone in a single point. The figure shows the case when that point is on the positive cone. If the point is on the negative cone, it is rejected as an intersection. Upper right: $\delta > 0$ and both roots are on the positive cone. Lower left: $\delta > 0$ and one root is on the positive cone. Lower right: $\delta > 0$ and neither point is on the positive cone.



In all cases, an implementation must test whether the quadratic roots lead to points on the positive cone. Also, the t-parameters of the linear component of intersection must be selected appropriately. The upper-right image shows that the intersection is a segment $\mathbf{P} + t\mathbf{U}$ for $t \in [t_0, t_1]$ where $t_1 > t_0$. The lower-left image shows that the intersection is a ray $\mathbf{P} + t\mathbf{U}$ for $t \in (-\infty, t_0)$.

If $c_2 = 0$, the vector $\mathbf{U}$ is a direction vector on the cone boundary because $|\mathbf{D} \cdot \mathbf{U}| = \cos(\theta)$. If $c_1 \neq 0$, the polynomial is in fact linear and has a single root $t = -c_0/c_1$. The line and double-sided cone have a single point of intersection. As before, we report the point as an intersection only when it is on the positive cone. If it is, we must choose the correct t-interval of intersection.

Although not needed in the computations, when $c_1 = 0$ define $\sigma = \text{Sign}(\mathbf{D} \cdot \mathbf{U})$ so that $\mathbf{D} \cdot \mathbf{U} = \sigma \cos(\theta)$. Observe that $\mathbf{U} \cdot (\sigma\mathbf{D} - \cos(\theta)\mathbf{U}) = 0$, which means $\sigma\mathbf{D} - \cos(\theta)\mathbf{U}$ is perpendicular to $\mathbf{U}$. The linear-term coefficient becomes

$$c_1 = (\mathbf{D} \cdot \mathbf{U})(\mathbf{D} \cdot \mathbf{\Delta}) - \cos^2(\theta)(\mathbf{U} \cdot \mathbf{\Delta}) = \cos(\theta)(\sigma\mathbf{D} - \cos(\theta)\mathbf{U}) \cdot \mathbf{\Delta}$$

The condition $c_1 = 0$ is equivalent to $\mathbf{P} = \mathbf{V}$ or $\mathbf{P} - \mathbf{V}$ is perpendicular to $\sigma\mathbf{D} - \cos(\theta)\mathbf{U}$.

If $c_2 = 0$ and $c_1 = 0$, the polynomial is the constant c_0. If this constant is not zero, the line and cone do not intersect. If this constant is zero, observe that

$$0 = c_0 = (\mathbf{D} \cdot \mathbf{\Delta})^2 - \cos^2(\theta)(\mathbf{\Delta} \cdot \mathbf{\Delta}) \tag{4}$$

which implies that $\mathbf{P}$ is on the double-sided cone boundary. We know that $\mathbf{U}$ is a direction vector on the double-sided cone boundary, so in fact the line $\mathbf{P} + t\mathbf{U}$ is entirely on the double-sided cone boundary. This implies the line contains $\mathbf{V}$. The t-interval of intersection is semiinfinite with the finite endpoint $t_0 = \mathbf{U} \cdot (\mathbf{P} - \mathbf{V})$. The infinite endpoint has sign determined by the sign of $\mathbf{D} \cdot \mathbf{U}$.

When the line intersects the positive cone, it does so with a t-interval $[t_0, t_1]$, where t_0 is finite or $-\infty$, t_1 is finite or $+\infty$, and $t_1 \geq t_0$ with equality possible (the line intersects the cone in a single point). We can further trim this interval for a finite positive cone where the constraints are $0 \leq \mathbf{D} \cdot (\mathbf{P} - \mathbf{V}) + t(\mathbf{D} \cdot \mathbf{U}) \leq h$. When $\mathbf{D} \cdot \mathbf{U} = 0$, no trimming is needed; otherwise, the height interval is $[h_0, h_1]$ where $h_0 = \min\{\mu_0, \mu_1\}$ and $h_1 = \max\{\mu_0, \mu_1\}$ with $\mu_0 = -\mathbf{D} \cdot \mathbf{\Delta}/\mathbf{D} \cdot \mathbf{U}$ and $\mu_1 = (h - \mathbf{D} \cdot \mathbf{\Delta})/\mathbf{D} \cdot \mathbf{U}$. The clamped t-interval of intersection is $[t_0, t_1] \cap [h_0, h_1]$, which possibly might be empty.

3 Intersection with a Ray

When the line does not intersect the cone, neither does the ray. When the line intersects the cone (finite or infinite), let the t-interval of intersection be $[t_0, t_1]$ with $t_1 \geq t_0$ and either endpoint possibly infinite in magnitude. The ray adds an additional constraint, the t-interval $[0, +\infty)$. The final candidate interval for the ray-cone intersection is $[t_0, t_1] \cap [0, +\infty)$, which can be semiinfinite, finite, or empty. An implementation must determine which of these is the case and report the appropriate intersection points.

4 Intersection with a Segment

When the line does not intersect the cone, neither does the segment. When the line intersects the cone (finite or infinite), let the t-interval of intersection be $[t_0, t_1]$ with $t_1 \geq t_0$ and either endpoint possibly infinite in magnitude. The segment adds an additional constraint. In the GTEngine implementation, we currently use a centered representation for the segment. The endpoints are $\mathbf{P}_0$ and $\mathbf{P}_1$. The centered representation is $\mathbf{X}(t) = \mathbf{C} + t\mathbf{U}$, where $\mathbf{C} = (\mathbf{P}_1 + \mathbf{P}_0)/2$, $\mathbf{U} = (\mathbf{P}_1 - \mathbf{P}_0)/|\mathbf{P}_1 - \mathbf{P}_0|$, $e = |\mathbf{P}_1 - \mathbf{P}_0|/2$, and $t \in [-e, e]$. The final candidate interval for the segment-cone intersection is $[t_0, t_1] \cap [-e, e]$, which can be finite or empty. An implementation must determine which of these is the case and report the appropriate intersection points.