Intersection of Linear and Circular Components in 2D

David Eberly, Geometric Tools, Redmond WA 98052
https://www.geometrictools.com/

Created: October 24, 2000
Last Modified: March 2, 2008

Contents

1 Introduction

A *line* in 2D is parameterized as $\mathbf{P} + t\mathbf{D}$, where $\mathbf{D}$ is a nonzero vector and where t is any real number. A *ray* is parameterized the same way, except that $t \geq 0$. The point $\mathbf{P}$ is the origin of the ray. A *segment* is also parameterized the same way, except that $0 \leq t \leq 1$. The points $\mathbf{P}$ and $\mathbf{P} + \mathbf{D}$ are the endpoints of the segment. A *linear component* is the general term for a line, a ray, or a segment.

A *circle* in 2D is represented by $|\mathbf{X} - \mathbf{C}|^2 = R^2$, where $\mathbf{C}$ is the center and $R > 0$ is the radius. The circle can be parameterized by $\mathbf{X}(\theta) = \mathbf{C} + R\mathbf{U}(\theta)$, where $\mathbf{U}(\theta) = (\cos\theta, \sin\theta)$ and where $0 \leq \theta < 2\pi$. An *arc* is parameterized the same way, except that $\theta_0 \leq \theta \leq \theta_1$, with $0 \leq \theta_0 < 2\pi$, $0 \leq \theta_1 < 4\pi$, and $\theta_0 < \theta_1$. The larger interval for θ_1 allows for arcs that intersect the positive x-axis. It is also possible to represent an arc by center $\mathbf{C}$, radius R, and two endpoints $\mathbf{A}$ and $\mathbf{B}$ that correspond to angles θ_0 and θ_1, respectively.

2 Intersection of Two Linear Components

Consider two lines $\mathbf{P}_0 + s\mathbf{D}_0$ and $\mathbf{P}_1 + t\mathbf{D}_1$ for real numbers s and t. They are either intersecting, nonintersecting and parallel, or the same line. To help determine which of these cases occurs, we use the *dot-perp* operator. Let $\mathbf{V}_0 = (x_0, y_0)$ and $\mathbf{V}_1 = (x_1, y_1)$. The *dot-perp* of the two vectors is

$$\mathbf{V}_0 \cdot \mathbf{V}_1^{\perp} = (x_0, y_0) \cdot (y_1, -x_1) = x_0 y_1 - x_1 y_0 \tag{1}$$

The superscript perpendicular symbol on $\mathbf{V}_1$ indicates that you swap the two components of $\mathbf{V}_1$ and change sign on the second component. The dot-perp operation has the property that $\mathbf{V}_0 \cdot \mathbf{V}_1^{\perp} = -\mathbf{V}_1 \cdot \mathbf{V}_0^{\perp}$.

A point of intersection, if any, can be found by solving the two equations in two unknowns implied by setting $\mathbf{P}_0 + s\mathbf{D}_0 = \mathbf{P}_1 + t\mathbf{D}_1$. Rearranging terms yields $s\mathbf{D}_0 - t\mathbf{D}_1 = \mathbf{P}_1 - \mathbf{P}_0$. Setting $\mathbf{\Delta} = \mathbf{P}_1 - \mathbf{P}_0$ and applying the dot-perp operation leads to

$$\mathbf{D}_0 \cdot \mathbf{D}_1^{\perp}\, s = \mathbf{\Delta} \cdot \mathbf{D}_1^{\perp}, \quad \mathbf{D}_0 \cdot \mathbf{D}_1^{\perp}\, t = \mathbf{\Delta} \cdot \mathbf{D}_0^{\perp} \tag{2}$$

If $\mathbf{D}_0 \cdot \mathbf{D}_1^{\perp} \neq 0$, then the lines intersect in a single point determined by

$$s = \frac{\mathbf{\Delta} \cdot \mathbf{D}_1^{\perp}}{\mathbf{D}_0 \cdot \mathbf{D}_1^{\perp}}, \quad t = \frac{\mathbf{\Delta} \cdot \mathbf{D}_0^{\perp}}{\mathbf{D}_0 \cdot \mathbf{D}_1^{\perp}} \tag{3}$$

If $\mathbf{D}_0 \cdot \mathbf{D}_1^{\perp} = 0$, then the lines are either nonintersecting and parallel or the same line. The directions must be parallel, so $\mathbf{D}_1 = c\mathbf{D}_0$ for some nonzero scalar c. The two conditions in Equation (2) reduce to the single condition

$$\mathbf{\Delta} \cdot \mathbf{D}_0 = 0 \tag{4}$$

This is a consistency condition. When true, the lines are the same. When false, the lines do not intersect.

The s- and t-values from Equation (3) correspond to the intersection of two lines. If you are working with rays and/or segments, the parameter domain constraints must be tested. For example, suppose that the second linear components is a ray. The domain constraint is $t \geq 0$. If the t-value of Equation (3) satisfies $t < 0$, then the line (corresponding to the s-value) and the ray (corresponding to the t-value) do not intersect.

Finally, if the lines containing the two linear components are the same line, the linear components intersect in a t-interval which is possibly empty or bounded or semiinfinite or infinite. Computing the interval of intersection is somewhat tedious, but not complicated. For example, suppose you have rays $\mathbf{P}_0 + s\mathbf{D}_0$ $(s \geq 0)$ and $\mathbf{P}_1 + t\mathbf{D}_1$ $(t \geq 0)$ that are on the same line. The point $\mathbf{P}_1$ is represented by $\mathbf{P}_1 = \mathbf{P}_0 + \bar{s}\mathbf{D}_0$, where $\bar{s} = \mathbf{D}_0 \cdot (\mathbf{P}_1 - \mathbf{P}_0)/\mathbf{D}_0 \cdot \mathbf{D}_0$. When $\mathbf{D}_0 \cdot \mathbf{D}_1 > 0$, the rays point in the same direction and necessarily overlap. The s-interval of overlap is $s \geq \max(\bar{s}, 0)$. When $\mathbf{D}_0 \cdot \mathbf{D}_1 < 0$, the rays point in opposite directions. There is no overlap when $\bar{s} < 0$. The overlap is at $\mathbf{P}_0$ when $\bar{s} = 0$. When $\bar{s} > 0$, the overlap is the s-interval $0 \leq s \leq \bar{s}$.

3 Intersection of a Linear and a Circular Component

Consider first a parameterized line $\mathbf{X}(t) = \mathbf{P} + t\mathbf{D}$ and a circle defined implicitly by $|\mathbf{X} - \mathbf{C}|^2 = R^2$. Substitute the line equation into the circle equation, define $\mathbf{\Delta} = \mathbf{P} - \mathbf{C}$, and obtain the quadratic equation in t:

$$|\mathbf{D}|^2 t^2 + 2(\mathbf{D} \cdot \mathbf{\Delta})t + |\mathbf{\Delta}|^2 - R^2 = 0 \tag{5}$$

The formal roots of the equation are

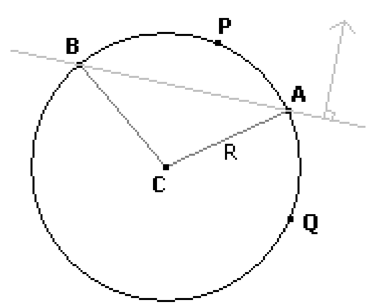
$$t = \frac{-\mathbf{D} \cdot \mathbf{\Delta} \pm \sqrt{(\mathbf{D} \cdot \mathbf{\Delta})^2 - |\mathbf{D}|^2(|\mathbf{\Delta}|^2 - R^2)}}{|\mathbf{D}|^2} \tag{6}$$

Define $\delta = (\mathbf{D} \cdot \mathbf{\Delta})^2 - |\mathbf{D}|^2(|\mathbf{\Delta}|^2 - R^2)$. If $\delta < 0$, the line does not intersect the circle. If $\delta = 0$, the line is tangent to the circle (one point of intersetion). If $\delta > 0$, the line intersects the circle in two points.

If the linear component is a ray, and if $\bar{t}$ is a real-valued root of the quadratic equation, then the corresponding point of intersection between line and circle is a point of intersection between ray and circle when $\bar{t} \geq 0$. Similarly, if the linear component is a segment, the line-circle point of intersection is also one for the segment and circle when $\bar{t} \in [0, 1]$.

If the circular component is an arc, the points of intersection between the linear component and circle must be tested to see if they are on the arc. Let the arc have endpoints $\mathbf{A}$ and $\mathbf{B}$, where the arc is that portion of the circle obtained by traversing the circle counterclockwise from $\mathbf{A}$ to $\mathbf{B}$. Notice that the line containing $\mathbf{A}$ and $\mathbf{B}$ separates the arc from the remainder of the circle. If $\mathbf{P}$ is a point on the circle, it is on the arc if and only if it is on the same side of that line as the arc. Figure 1 illustrates this.

Figure 1. An arc of a circle spanned counterclockwise from $\mathbf{A}$ to $\mathbf{B}$. The line containing $\mathbf{A}$ and $\mathbf{B}$ separates the circle into the arc itself and the remainder of the circle. Point $\mathbf{P}$ is on the arc since it is on the same side of the line as the arc. Point $\mathbf{Q}$ is not on the arc since it is on the opposite side of the line.



The algebraic condition for the circle point $\mathbf{P}$ to be on the arc is

$$(\mathbf{P}-\mathbf{A})\cdot(\mathbf{B}-\mathbf{A})^{\perp}\geq 0 \tag{7}$$

4 Intersection of Circular Components

Let the two circles be represented by $|\mathbf{X}-\mathbf{C}_i|^2 = R_i^2$ for $i = 0, 1$. The points of intersection, if any, are determined by the following construction. Define $\mathbf{U} = \mathbf{C}_1 - \mathbf{C}_0 = (u_0, u_1)$. Define $\mathbf{V} = \mathbf{U}^{\perp} = (u_1, -u_0)$. Notice that $|\mathbf{U}|^2 = |\mathbf{V}|^2 = |\mathbf{C}_1 - \mathbf{C}_0|^2$ and $\mathbf{U}\cdot\mathbf{V} = 0$. The intersection points can be written in the form

$$\mathbf{X} = \mathbf{C}_0 + s\mathbf{U} + t\mathbf{V} = \mathbf{C}_1 + (s-1)\mathbf{U} + t\mathbf{V} \tag{8}$$

Substituting $\mathbf{X} = \mathbf{C}_0 + s\mathbf{U} + t\mathbf{V}$ into the circle equation $|\mathbf{X}-\mathbf{C}_0|^2 = R_0^2$ yields

$$(s^2+t^2)|\mathbf{U}|^2 = R_0^2 \tag{9}$$

Substituting $\mathbf{X} = \mathbf{C}_1 + (s-1)\mathbf{U} + t\mathbf{V}$ into the circle equation $|\mathbf{X}-\mathbf{C}_1|^2 = R_1^2$ yields

$$((s-1)^2+t^2)|\mathbf{U}|^2 = R_1^2 \tag{10}$$

Subtracting the Equations (9) and (10) and solving for s yields

$$s = \frac{1}{2}\left(\frac{R_0^2 - R_1^2}{|\mathbf{U}|^2} + 1\right) \tag{11}$$

Replacing this in the Equation (9) and solving for t^2 yields

$$t^2 = \frac{R_0^2}{|\mathbf{U}|^2} - s^2 = \frac{-(|\mathbf{U}|^2 - (R_0+R_1)^2)(|\mathbf{U}|^2 - (R_0-R_1)^2)}{4|\mathbf{U}|^2} \tag{12}$$

In order for there to be solutions, the right-hand side of Equation (12) must be nonnegative. The numerator of the fraction must be nonnegative, which leads to the existence condition

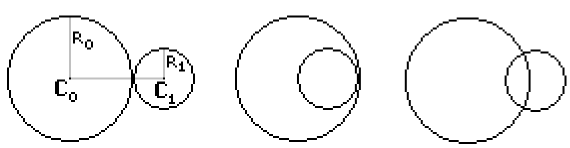
$$(|\mathbf{U}|^2 - (R_0+R_1)^2)(|\mathbf{U}|^2 - (R_0-R_1)^2) \leq 0 \tag{13}$$

This in turn can be reduced to

$$|R_0 - R_1| \leq |\mathbf{U}| \leq |R_0 + R_1|. \tag{14}$$

If $|\mathbf{U}| = |R_0 + R_1|$, then each circle is outside the other circle, but just tangent. The point of intersection is $\mathbf{C}_0 + (R_0/(R_0+R_1))\mathbf{U}$. If $|\mathbf{U}| = |R_0 - R_1|$, then the circles are nested and just tangent. The circles are the same if $|\mathbf{U}| = 0$ and $R_0 = R_1$, otherwise the point of intersection is $\mathbf{C}_0 + (R_0/(R_0-R_1))\mathbf{U}$. If $|R_0 - R_1| < |\mathbf{U}| < |R_0 + R_1|$, then the two circles intersect in two points. Figure 2 shows the various circle-circle configurations.

Figure 2. Relationship of two circles, where $\mathbf{U} = \mathbf{C}_1 - \mathbf{C}_0$. Left: $|\mathbf{U}| = |R_0 + R_1|$. Middle: $|\mathbf{U}| = |R_0 - R_1|$. Right: $|R_0 - R_1| < |\mathbf{U}| < |R_0 + R_1|$.



In Figure 2, the intersection points for the left and middle configurations are $\mathbf{C}_0 + s\mathbf{U}$, where s is defined in Equation (11). The intersection points for the right configuration are $\mathbf{C}_0 + s\mathbf{U} \pm t\mathbf{V}$, where t is the positive square root obtained from Equation (12), namely,

$$t = \sqrt{\frac{R_0^2}{|\mathbf{U}|^2} - s^2} \tag{15}$$

If either or both circular components are arcs, the circle-circle points of intersection must be tested if they are on the arc (or arcs) using the circular-point-on-arc test of Equation (7).