Intersection of a Cylinder and a Plane

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1 Representation of a Plane

A plane contains a point $\boldsymbol{P}$ and has unit-length normal vector $\boldsymbol{N}$. A point $\boldsymbol{X}$ is on the plane whenever

$$\boldsymbol{N} \cdot (\boldsymbol{X} - \boldsymbol{P}) = 0 \tag{1}$$

This algebraic equation states that the vector $\boldsymbol{X} - \boldsymbol{P}$ is perpendicular to $\boldsymbol{N}$. The plane is parameterized by

$$\boldsymbol{X}(\alpha, \beta) = \boldsymbol{P} + \alpha \boldsymbol{A} + \beta \boldsymbol{B} \tag{2}$$

where $\boldsymbol{N}$, $\boldsymbol{A}$ and $\boldsymbol{B}$ are unit-length vectors for which $\{\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{N}\}$ is a right-handed orthonormal basis. The parameters α and β are any real numbers.

2 Representation of a Cylinder

The cylinder is considered to be hollow rather than a solid, but the distinction is irrelevant for intersection queries. The definition here assumes the cylinder is hollow. The cylinder axis contains the point $\boldsymbol{C}$, which I call the cylinder center, and the axis has unit-length direction $\boldsymbol{W}$. The cylinder consists of those points at a distance $r > 0$ from the cylinder axis. For each cylinder point $\boldsymbol{X}$, the distance to the axis is the length of the projection onto the plane $\boldsymbol{W} \cdot (\boldsymbol{X} - \boldsymbol{C})$. The projection is

$$\boldsymbol{Y} = (\boldsymbol{X} - \boldsymbol{C}) - (\boldsymbol{W} \cdot (\boldsymbol{X} - \boldsymbol{C})) = \left(I - \boldsymbol{W}\boldsymbol{W}^{\mathsf{T}}\right)(\boldsymbol{X} - \boldsymbol{C}) \tag{3}$$

where I is the 3×3 identity matrix. The vector $\boldsymbol{W}$ is 3×1 and the vector $\boldsymbol{W}^{\mathsf{T}}$ is 1×3, so the product $\boldsymbol{W}\boldsymbol{W}^{\mathsf{T}}$ is a 3×3 matrix. The length of the projection is $|\boldsymbol{Y}| = r$, so the squared length is an algebraic equation that represents the cylinder,

$$(\boldsymbol{X} - \boldsymbol{C})^{\mathsf{T}} \left(I - \boldsymbol{W}\boldsymbol{W}^{\mathsf{T}}\right)(\boldsymbol{X} - \boldsymbol{C}) = r^2 \tag{4}$$

The cylinder is parameterized by

$$\boldsymbol{X}(\theta, s) = \boldsymbol{C} + (r\cos\theta)\boldsymbol{U} + (r\sin\theta)\boldsymbol{V} + s\boldsymbol{W} \tag{5}$$

where $\boldsymbol{U}$, $\boldsymbol{V}$ and $\boldsymbol{W}$ are unit-length vectors that are mutually perpendicular, and $\theta \in [0, 2\pi)$.

The parameter $s \in [-h/2, +h/2]$ for a *finite cylinder*, where $h \in (0, +\infty)$ is the height of the cylinder.

The parameter $s \in (-\infty, +\infty)$ for an *infinite cylinder*. When computing with floating-point numbers of type T that is float or double, it is possible to represent infinity using std::numeric_limits<T>::infinity(). This is a problem, however, when computing with arbitrary-precision numbers that do not have a representation for infinity. In particular, the BSNumber and BSRational types in the Geometric Tools code do not have such a representation. To remedy this, one could use the maximum value of the type, say, std::numeric_limits<T>::max(), which allows BSNumber and BSRational to convert to arbitrary precision. The problem is that this representation is finite, so the infinite cylinder would be treated as if it were a finite cylinder.

To avoid these problems, the Geometric Tools Cylinder class chooses to identify an infinite cylinder by setting the height member to -1. This requires some extra logic in algorithms that require comparing heights, but it does allow for quickly determining whether a cylinder is finite or infinite.

3 Test-Intersection Queries

A *test-intersection query* determines whether two objects intersect. The actual set of intersection (if any) is not computed. In some applications, knowing that two objects intersect is sufficient, so you avoid the computational cost of actually computing the set of intersection.

3.1 Query for an Infinite Cylinder

If the cylinder axis is not parallel to the plane, the cylinder and plane must intersect. The test for non-parallel is simply $\boldsymbol{N} \cdot \boldsymbol{W} \neq \mathbf{0}$.

If the cylinder axis is parallel to the plane, a condition specified by $\boldsymbol{N} \cdot \boldsymbol{W} = \mathbf{0}$, the cylinder and plane intersect when the cylinder radius is larger or equal to the distance from the cylinder axis to the plane. It is sufficient to compute the distance from the cylinder center to the plane and compare to the radius,

$$|\boldsymbol{N} \cdot (\boldsymbol{C} - \boldsymbol{P})| \leq r \tag{6}$$

The quantity $\boldsymbol{N} \cdot (\boldsymbol{C} - \boldsymbol{P})$ is the projection of $(\boldsymbol{C} - \boldsymbol{P})$ onto the normal axis, so it is the signed distance from $\boldsymbol{C}$ to the plane. The absolute value changes this to the distance.

Listing 1 contains pseudocode for the test-intersection query when $\boldsymbol{N}$ and $\boldsymbol{W}$ are unit-length vectors.

Listing 1. The test-intersection query for an infinite cylinder and a plane when $\boldsymbol{N}$ and $\boldsymbol{W}$ are unit-length vectors. The function returns true if and only if the infinite cylinder and plane intersect.

```
// N and W are required to be unit length.
bool TestIntersection(Vector3<T> P, Vector3<T> N, Vector3<T> C, Vector3<T> W, T r)
{
    if (Dot(N, W) != 0)
    {
        // The cylinder axis and plane are not parallel.
        return true;
    }
    else
    {
        // The cylinder axis and plane are parallel.
        return abs(Dot(N, C − P)) <= r;
    }
}
```

3.2 Query for a Finite Cylinder

The test-intersection query for a finite cylinder is more complicated than that for an infinite cylinder. Geometrically, consider the normal line $\boldsymbol{P} + t\boldsymbol{N}$ for any real number t. The projection of the plane onto the normal line is $t = 0$. The projection of the cylinder onto the plane is a t-interval, say, $[t_{\min}, t_{\max}]$. The cylinder and the plane intersect when $0 \in [t_{\min}, t_{\max}]$.

The projections of the cylinder points $\boldsymbol{X}(\theta, s)$ onto the normal line are

$$\boldsymbol{N} \cdot (\boldsymbol{X}(\theta, s) - \boldsymbol{P}) = \boldsymbol{N} \cdot (\boldsymbol{C} - \boldsymbol{P}) + r((\cos\theta)\boldsymbol{N} \cdot \boldsymbol{U} + (\sin\theta)\boldsymbol{N} \cdot \boldsymbol{V}) + s\boldsymbol{N} \cdot \boldsymbol{W} \tag{7}$$

The t-interval of projection has endpoints

$$t_{\min} = \min_{(\theta,s)}(\boldsymbol{N} \cdot (\boldsymbol{X}(\theta, s) - \boldsymbol{P})), \quad t_{\max} = \min_{(\theta,s)}(\boldsymbol{N} \cdot (\boldsymbol{X}(\theta, s) - \boldsymbol{P})), \tag{8}$$

The maximum value of $s\boldsymbol{N} \cdot \boldsymbol{W}$ is $(h/2)|\boldsymbol{N} \cdot \boldsymbol{W}|$. The maximum value of $(\cos\theta)\boldsymbol{N} \cdot \boldsymbol{U} + (\sin\theta)\boldsymbol{N} \cdot \boldsymbol{V}$ is achieved when $(\cos\theta, \sin\theta)$ is in the same direction as $(\boldsymbol{N} \cdot \boldsymbol{U}, \boldsymbol{N} \cdot \boldsymbol{V})$. This implies

$$(\cos\theta, \sin\theta) = \frac{(\boldsymbol{N} \cdot \boldsymbol{U}, \boldsymbol{N} \cdot \boldsymbol{V})}{\sqrt{(\boldsymbol{N} \cdot \boldsymbol{U})^2 + (\boldsymbol{N} \cdot \boldsymbol{V})^2}} \tag{9}$$

in which case

$$(\cos\theta)\boldsymbol{N} \cdot \boldsymbol{U} + (\sin\theta)\boldsymbol{N} \cdot \boldsymbol{V} = \sqrt{(\boldsymbol{N} \cdot \boldsymbol{U})^2 + (\boldsymbol{N} \cdot \boldsymbol{V})^2} = \sqrt{1 - (\boldsymbol{N} \cdot \boldsymbol{W})^2} \tag{10}$$

The last equality is true because $\boldsymbol{N}$ can be represented in the orthonormal basis $\{\boldsymbol{U}, \boldsymbol{V}, \boldsymbol{W}\}$ by

$$\boldsymbol{N} = (\boldsymbol{N} \cdot \boldsymbol{U})\boldsymbol{U} + (\boldsymbol{N} \cdot \boldsymbol{V})\boldsymbol{V} + (\boldsymbol{N} \cdot \boldsymbol{W})\boldsymbol{W} \tag{11}$$

The 3-tuple of coefficients in the representation must form a unit-length vector,

$$1 = \boldsymbol{N} \cdot \boldsymbol{N} = (\boldsymbol{N} \cdot \boldsymbol{U})^2 + (\boldsymbol{N} \cdot \boldsymbol{V})^2 + (\boldsymbol{N} \cdot \boldsymbol{W})^2 \tag{12}$$

The construction implies

$$\begin{array}{rcl} t_{\min} & = & \boldsymbol{N} \cdot (\boldsymbol{C} - \boldsymbol{P}) - r\sqrt{1 - (\boldsymbol{N} \cdot \boldsymbol{W})^2} - (h/2)|\boldsymbol{N} \cdot \boldsymbol{W}| \\ t_{\max} & = & \boldsymbol{N} \cdot (\boldsymbol{C} - \boldsymbol{P}) + r\sqrt{1 - (\boldsymbol{N} \cdot \boldsymbol{W})^2} + (h/2)|\boldsymbol{N} \cdot \boldsymbol{W}| \end{array} \tag{13}$$

The condition $0 \in [t_{\min}, t_{\max}]$ is equivalent to $t_{\min} \leq 0 \leq t_{\max}$ which in turn is equivalent to

$$|\boldsymbol{N} \cdot (\boldsymbol{C} - \boldsymbol{P})| \leq r\sqrt{1 - (\boldsymbol{N} \cdot \boldsymbol{W})^2} + (h/2)|\boldsymbol{N} \cdot \boldsymbol{W}| = r|\boldsymbol{N} \times \boldsymbol{W}| + (h/2)|\boldsymbol{N} \cdot \boldsymbol{W}| \tag{14}$$

When the cylinder axis is parallel to the plane, $\boldsymbol{N} \cdot \boldsymbol{W} = 0$ and equation (14) reduces to equation (6).

Listing 2 contains pseudocode for the test-intersection query when $\boldsymbol{N}$ and $\boldsymbol{W}$ are unit-length vectors.

Listing 2. The test-intersection query for a finite cylinder and a plane when $\boldsymbol{N}$ and $\boldsymbol{W}$ are unit-length vectors. The function returns true if and only if the finite cylinder and plane intersect.

```
// N and W are required to be unit length.
bool TestIntersection(Vector3<T> P, Vector3<T> N, Vector3<T> C, Vector3<T> W, T r, T h)
{
    T absDotNCmP = abs(Dot(N, C − P));
    T absDotNW = abs(Dot(N, W));
    return absDotNCmP <= r * sqrt(1 − absDotNW * absDotNW) + (h / 2) * absDotNW;
}
```

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4 Find-Intersection Queries

A *find-intersection query* determines the set of intersection of two objects, which generally requires more computation time than a test-intersection query.

4.1 Query for an Infinite Cylinder

4.1.1 Cylinder Axis Parallel to the Plane

If the cylinder intersects the plane, the set of intersection is either a single line when the plane is tangent to the cylinder or two lines when the plane cuts through the cylinder. In either case the line direction is $\boldsymbol{W}$. The set $\{\boldsymbol{N}, \boldsymbol{W}, \boldsymbol{N} \times \boldsymbol{W}\}$ is a right-handed orthonormal basis.

When the plane is tangent to the cylinder, the point on the intersection line closest to $\boldsymbol{C}$ is $\boldsymbol{K}$, where $\boldsymbol{C} - \boldsymbol{K}$ is parallel to $\boldsymbol{N}$ and $|\boldsymbol{C} - \boldsymbol{K}| = r$. Specifically, $\boldsymbol{K} = \boldsymbol{C} - d\boldsymbol{N}$, where $d = \boldsymbol{N} \cdot (\boldsymbol{C} - \boldsymbol{P})$ is the signed distance from $\boldsymbol{C}$ to the plane which in this case satisfies $|d| = r$.

When the plane cuts through the cylinder, two points on the intersection lines are the projection of $\boldsymbol{C}$ onto the plane offset by a distance $\ell = \sqrt{r^2 - d^2}$ in the directions $\pm \boldsymbol{N} \times \boldsymbol{W}$, where d is the quantity defined in the previous paragraph. It is the case that $r^2 > d^2$. The points are $\boldsymbol{K} = \boldsymbol{C} - d\boldsymbol{N} \pm \ell \boldsymbol{N} \times \boldsymbol{W}$. Observe that when $\ell = 0$, the two lines are the same, which is the case when the plane is tangent to the cylinder.

4.1.2 Cylinder Axis Not Parallel to the Plane

The cylinder and plane intersect in a circle or an ellipse. A right-handed orthonormal basis is computed from the plane normal, namely, $\{\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{N}\}$. The first two basis vectors are used for the parametric representation in equation (2).

Substituting equation (2) into equation (4) and defining $\boldsymbol{\Delta} = \boldsymbol{P} - \boldsymbol{C}$ and $M = I - \boldsymbol{W}\boldsymbol{W}^{\mathsf{T}}$ leads to the equation $(\alpha\boldsymbol{A} + \beta\boldsymbol{B} + \boldsymbol{\Delta})^{\mathsf{T}} M (\alpha\boldsymbol{A} + \beta\boldsymbol{B} + \boldsymbol{\Delta}) = r^2$. Expanding this, we obtain the quadratic equation in α and β,

$$\begin{aligned} 0 &= (\boldsymbol{A}^{\mathsf{T}} M \boldsymbol{A})\alpha^2 + 2(\boldsymbol{A}^{\mathsf{T}} M \boldsymbol{B})\alpha\beta + (\boldsymbol{B}^{\mathsf{T}} M \boldsymbol{B})\beta^2 + 2(\boldsymbol{A}^{\mathsf{T}} M \boldsymbol{\Delta})\alpha + 2(\boldsymbol{B}^{\mathsf{T}} M \boldsymbol{\Delta})\beta + (\boldsymbol{\Delta}^{\mathsf{T}} M \boldsymbol{\Delta} - r^2) \\ &= \begin{bmatrix} \alpha & \beta \end{bmatrix} Q_2 \begin{bmatrix} \alpha \\ \beta \end{bmatrix} + \begin{bmatrix} \alpha & \beta \end{bmatrix} \boldsymbol{Q}_1 + q_0 \\ &= \boldsymbol{\xi}^{\mathsf{T}} Q_2 \boldsymbol{\xi} + \boldsymbol{Q}_1^{\mathsf{T}} \boldsymbol{\xi} + q_0 \end{aligned} \tag{15}$$

where

$$\boldsymbol{\xi} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}, \ Q_2 = \begin{bmatrix} \boldsymbol{A}^{\mathsf{T}} M \boldsymbol{A} & \boldsymbol{A}^{\mathsf{T}} M \boldsymbol{B} \\ \boldsymbol{A}^{\mathsf{T}} M \boldsymbol{B} & \boldsymbol{B}^{\mathsf{T}} M \boldsymbol{B} \end{bmatrix}, \ \boldsymbol{Q}_1 = 2 \begin{bmatrix} \boldsymbol{A}^{\mathsf{T}} M \boldsymbol{\Delta} \\ \boldsymbol{B}^{\mathsf{T}} M \boldsymbol{\Delta} \end{bmatrix}, \ q_0 = \boldsymbol{\Delta}^{\mathsf{T}} M \boldsymbol{\Delta} - r^2 \tag{16}$$

The quadratic equation can be factored to $(\boldsymbol{\xi} - \boldsymbol{k})^{\mathsf{T}} S (\boldsymbol{\xi} - \boldsymbol{k}) = 1$, where $\boldsymbol{k} = [k_0 \, k_1]^{\mathsf{T}}$ is a 2×1 vector and S is a 2×2 positive definite matrix.

The first step is to complete the square of the quadratic equation. Define $\boldsymbol{k} = -Q_2^{-1} \boldsymbol{Q}_1 / 2$; then

$$\begin{aligned} 0 &= \boldsymbol{\xi}^{\mathsf{T}} Q_2 \boldsymbol{\xi} + \boldsymbol{Q}_1^{\mathsf{T}} \boldsymbol{\xi} + q_0 \\ &= (\boldsymbol{\xi} - \boldsymbol{k})^{\mathsf{T}} Q_2 (\boldsymbol{\xi} - \boldsymbol{k}) + (\boldsymbol{Q}_1 + 2Q_2\boldsymbol{k})^{\mathsf{T}} \boldsymbol{\xi} + \left(q_0 - \boldsymbol{k}^{\mathsf{T}} Q_2 \boldsymbol{k}\right) \\ &= (\boldsymbol{\xi} - \boldsymbol{k})^{\mathsf{T}} Q_2 (\boldsymbol{\xi} - \boldsymbol{k}) + \left(q_0 - \boldsymbol{k}^{\mathsf{T}} Q_2 \boldsymbol{k}\right) \end{aligned} \tag{17}$$

The equation can be simplified to

$$1 = (\boldsymbol{\xi} - \boldsymbol{k})^{\mathsf{T}} \left(\frac{Q_2}{\boldsymbol{k}^{\mathsf{T}} Q_2 \boldsymbol{k} - q_0} \right) (\boldsymbol{\xi} - \boldsymbol{k}) = (\boldsymbol{\xi} - \boldsymbol{k})^{\mathsf{T}} S (\boldsymbol{\xi} - \boldsymbol{k}) \tag{18}$$

where the second equality defines S.

The second step is to use an eigendecomposition $S = RDR^{\mathsf{T}}$ where

$$R = \begin{bmatrix} r_{00} & r_{01} \\ r_{10} & r_{11} \end{bmatrix} = \begin{bmatrix} \boldsymbol{v}_0 & \boldsymbol{v}_1 \end{bmatrix}, \; D = \begin{bmatrix} (1/e_0)^2 & 0 \\ 0 & (1/e_1)^2 \end{bmatrix} \tag{19}$$

The columns of R are the eigenvectors $\boldsymbol{v}_0$ and $\boldsymbol{v}_1$ of S. The diagonal elements of D are the corresponding eigenvalues of S. This leads to

$$\begin{aligned} 1 &= (\boldsymbol{\xi} - \boldsymbol{k})^{\mathsf{T}} RDR^{\mathsf{T}} (\boldsymbol{\xi} - \boldsymbol{k}) \\ &= (\boldsymbol{\xi} - \boldsymbol{k})^{\mathsf{T}} \left(\frac{\boldsymbol{v}_0 \boldsymbol{v}_0^{\mathsf{T}}}{e_0^2} + \frac{\boldsymbol{v}_1 \boldsymbol{v}_1^{\mathsf{T}}}{e_1^2} \right) (\boldsymbol{\xi} - \boldsymbol{k}) \\ &= \left(\frac{\boldsymbol{v}_0 \cdot (\boldsymbol{\xi} - \boldsymbol{k})}{e_0} \right)^2 + \left(\frac{\boldsymbol{v}_1 \cdot (\boldsymbol{\xi} - \boldsymbol{k})}{e_1} \right)^2 \end{aligned} \tag{20}$$

The ellipse center is $\boldsymbol{k}$ and has axis directions $\boldsymbol{v}_0$ and $\boldsymbol{v}_1$ with corresponding extents e_0 and e_1. A parameterization is $\boldsymbol{\xi} = \boldsymbol{k} + \eta_0 \boldsymbol{v}_0 + \eta_1 \boldsymbol{v}_1$ where $(\eta_0/e_0)^2 + (\eta_1/e_1)^2 = 1$, the last equation the standard form of an axis-aligned ellipse with center at the origin and extents e_0 and e_1.

The $\boldsymbol{\xi}$ are the coordinates within the plane relative to the choice of vectors $\boldsymbol{A}$ and $\boldsymbol{B}$. The 2D ellipse needs to be lifted to a 3D ellipse living in the plane. The 3D ellipse center is $\boldsymbol{P} + k_0 \boldsymbol{A} + k_1 \boldsymbol{B}$. The 3D ellipse axis directions are $r_{00} \boldsymbol{A} + r_{10} \boldsymbol{B}$ and $r_{01} \boldsymbol{A} + r_{11} \boldsymbol{B}$. The corresponding extents are e_0 and e_1.

4.2 Query for a Finite Cylinder

The test-intersection query for a finite cylinder and a plane is executed first. If there is an intersection, the find-intersection query is then executed.

The trim lines are the intersection of the plane $\boldsymbol{N} \cdot (\boldsymbol{X} - \boldsymbol{P}) = 0$ and the cylinder end planes $\boldsymbol{W} \cdot (\boldsymbol{X} - \boldsymbol{C}) = \pm h/2$. The trimmed ellipse lives in the infinite rectangular strip defined by $-h/2 \leq \boldsymbol{W} \cdot (\boldsymbol{X} - \boldsymbol{C}) \leq h/2$.

In the Geometric Tools implementation, the intersection of the plane and the infinite cylinder are computed. The trim lines are also computed. All are returned by the query. The caller can use these as desired.

4.2.1 Cylinder Axis Parallel to the Plane

If the cylinder axis is parallel to the plane, the line or strip (region bounded by two lines) of intersection must be trimmed by the intersection of the cylinder end planes to obtain a segment or rectangle. The trim lines are perpendicular to the projection of the cylinder axis onto the plane; the unnormalized direction is $\boldsymbol{N} \times \boldsymbol{W}$.

4.2.2 Cylinder Axis Not Parallel to the Plane

If the cylinder axis is not parallel to the plane, the ellipse of intersection is computed. The ellipse must be trimmed by the lines of intersection between the cylinder end planes and the input plane. The unnormalized direction of the trim lines is $\boldsymbol{N} \times \boldsymbol{W}$ and is that of the minor axis of the ellipse. The unnormalized major axis direction is $\widehat{\boldsymbol{A}} = \left(I - \boldsymbol{N}\boldsymbol{N}^{\mathsf{T}}\right)\boldsymbol{W}$, which says geometrically that this direction is the projection of the cylinder direction onto the plane. The unnormalized minor axis direction is $\widehat{\boldsymbol{B}} = \boldsymbol{N} \times \widehat{\boldsymbol{A}} = \boldsymbol{N} \times \boldsymbol{W}$.