Intersection of an Infinite Cylinder and a Plane

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Contents

1 Representation of a Plane

A plane contains a point $\mathbf{P}$ and has unit-length normal vector $\mathbf{N}$. A point $\mathbf{X}$ is on the plane whenever

$$\mathbf{N} \cdot (\mathbf{X} - \mathbf{P}) = 0 \tag{1}$$

This algebraic equation states that the vector $\mathbf{X} - \mathbf{P}$ is perpendicular to $\mathbf{N}$. The plane is parameterized by

$$\mathbf{X}(\alpha, \beta) = \mathbf{P} + \alpha\mathbf{A} + \beta\mathbf{B} \tag{2}$$

where $\mathbf{N}$, $\mathbf{A}$, and $\mathbf{B}$ are unit-length vectors that are mutually perpendicular. The parameters α and β are any real numbers.

2 Representation of an Infinite Cylinder

The cylinder axis contains the point $\mathbf{C}$ and has unit-length direction $\mathbf{D}$. The cylinder consists of those points at a distance r from the cylinder axis. An algebraic equation that represents the cylinder is derived as follows.

To project out the $\mathbf{D}$ portion of a vector $\mathbf{V}$, you compute

$$\mathbf{V}' = \mathbf{V} - (\mathbf{D}^{\mathsf{T}}\mathbf{V})\mathbf{D} = I\mathbf{V} - \mathbf{D}(\mathbf{D}^{\mathsf{T}}\mathbf{V}) = (I - \mathbf{D}\mathbf{D}^{\mathsf{T}})\mathbf{V}$$

where I is the identity matrix and where the superscript T denotes the transpose operator. The matrix $M = I - \mathbf{D}\mathbf{D}^{\mathsf{T}}$ is said to be a projection matrix. The vector $\mathbf{V}' = M\mathbf{V}$ is in the plane containing the origin and that is perpendicular to $\mathbf{D}$. A projection matrix has the property that if you apply it twice, you get the same thing as if you applied it once; that is, the projection of a projected vector does not change the projected vector. Mathematically, this means $M^2 = M$. A projection matrix is also symmetric, $M^{\mathsf{T}} = M$.

If $\mathbf{X}$ is any point on the cylinder, the projection of the vector $\mathbf{V} = \mathbf{X} - \mathbf{C}$ onto a plane perpendicular to $\mathbf{D}$ must have length r, in which case $r = |M\mathbf{V}|$. The squared equation is $r^2 = |M\mathbf{V}|^2 = (M\mathbf{V})^{\mathsf{T}}(M\mathbf{V}) = \mathbf{V}^{\mathsf{T}}M^T M\mathbf{V} = \mathbf{V}^{\mathsf{T}}M^2\mathbf{V} = \mathbf{V}^{\mathsf{T}}M\mathbf{V} = (\mathbf{X} - \mathbf{C})^{\mathsf{T}}M(\mathbf{X} - \mathbf{C})$. Thus, the algebraic equation is the quadratic equation

$$(\mathbf{X} - \mathbf{C})^{\mathsf{T}}(I - \mathbf{D}\mathbf{D}^{\mathsf{T}})(\mathbf{X} - \mathbf{C}) = r^2 \tag{3}$$

The cylinder is parameterized by

$$\mathbf{X}(s, \theta) = \mathbf{C} + s\mathbf{D} + (r\cos\theta)\mathbf{U} + (r\sin\theta)\mathbf{V} \tag{4}$$

where $\mathbf{D}$, $\mathbf{U}$, and $\mathbf{V}$ are unit-length vectors that are mutually perpendicular. The parameter s is any real number and the parameter $\theta \in [0, 2\pi)$.

3 Intersection of the Objects

I assume here that the cylinder axis is not parallel to the plane, so your geometric intuition should convince you that the intersection of the cylinder and the plane is an ellipse.

Substituting equation (2) into equation (3), defining $\mathbf{\Delta} = \mathbf{P} - \mathbf{C}$, and defining $M = I - \mathbf{D}\mathbf{D}^\mathsf{T}$, leads to $(\mathbf{\Delta} + \alpha\mathbf{A} + \beta\mathbf{B})^\mathsf{T} M(\mathbf{\Delta} + \alpha\mathbf{A} + \beta\mathbf{B}) = r^2$. Expanding this, we obtain the quadratic equation in α and β,

$$(\mathbf{A}^\mathsf{T} M\mathbf{A})\alpha^2 + 2(\mathbf{A}^\mathsf{T} M\mathbf{B})\alpha\beta + (\mathbf{B}^\mathsf{T} M\mathbf{B})\beta^2 + 2(\mathbf{A}^\mathsf{T} M\mathbf{\Delta})\alpha + 2(\mathbf{B}^\mathsf{T} M\mathbf{\Delta})\beta + (\mathbf{\Delta}^\mathsf{T} M\mathbf{\Delta} - r^2) = 0 \tag{5}$$

The factoring of a general quadratic equation is a topic discussed in text books on analytic geometry.

$$\left(\frac{\alpha - \alpha_0}{L_\alpha}\right)^2 + \left(\frac{\beta - \beta_0}{L_\beta}\right)^2 = 1 \tag{6}$$

where the ellipse center is (α_0, β_0) and the axis half-lengths are L_α and L_β.

The (α, β) values are the coordinates within the plane relative to the choice of vectors $\mathbf{A}$ and $\mathbf{B}$. Rather than work through the details of factoring equation (5) for arbitrary choices of $\mathbf{A}$ and $\mathbf{B}$, let us choose a pair of vectors for which the factorization is immediate.

Specifically, your intuition should convince you that the major axis of the ellipse is in the direction of the projection of the cylinder axis onto the plane. That is, we need to project out the $\mathbf{N}$ component from $\mathbf{D}$. Choose $\mathbf{A}$ to be a unit-length vector in the projected direction:

$$\mathbf{A} = \frac{(I - \mathbf{N}\mathbf{N}^\mathsf{T})\mathbf{D}}{|(I - \mathbf{N}\mathbf{N}^\mathsf{T})\mathbf{D}|} \tag{7}$$

The vector $\mathbf{B}$ is chosen to be a unit-length vector that is perpendicular to both $\mathbf{N}$ and $\mathbf{A}$, namely,

$$\mathbf{B} = \mathbf{N} \times \mathbf{A} = \frac{\mathbf{N} \times \mathbf{D}}{|\mathbf{N} \times \mathbf{D}|} \tag{8}$$

Then $\mathbf{A}^\mathsf{T} M\mathbf{B} = -(\mathbf{D} \cdot \mathbf{A})(\mathbf{D} \cdot \mathbf{B}) = 0$. Define $c_0 = \mathbf{A}^\mathsf{T} M\mathbf{A}$, $c_1 = \mathbf{B}^\mathsf{T} M\mathbf{B}$, $c_2 = 2\mathbf{A}^\mathsf{T} M\mathbf{\Delta}$, $c_3 = 2\mathbf{B}^\mathsf{T} M\mathbf{\Delta}$, and $c_4 = \mathbf{\Delta}^\mathsf{T} M\mathbf{\Delta} - r^2$. Equation (5) is then

$$c_0\alpha^2 + c_1\beta^2 + c_2\alpha + c_3\beta + c_4 = 0$$

Completing the square on the α terms and similarly on the β terms leads to

$$\frac{(\alpha + c_2/(2c_0))^2}{c_1} + \frac{(\beta + c_3/(2c_1))^2}{c_0} = \frac{c_2^2/(4c_0) + c_3^2/(4c_1) - c_4}{c_0 c_1} = \lambda \tag{9}$$

where the last equality defines λ. Matching this to equation (6), we have

$$\alpha_0 = -c_2/(2c_0), \quad \beta_0 = -c_3/(2c_1), \quad L_\alpha = \sqrt{c_1\lambda}, \quad L_\beta = \sqrt{c_0\lambda}$$

If you so choose, you may parameterize this ellipse using $\alpha = \alpha_0 + L_\alpha \cos\phi$, $\beta = \beta_0 + L_\beta \sin\phi$, and substitute into equation (2) to obtain

$$\mathbf{X}(\alpha, \beta) = (\mathbf{P} + \alpha_0\mathbf{A} + \beta_0\mathbf{B}) + (L_\alpha \cos\phi)\mathbf{A} + (L_\beta \sin\phi)\mathbf{B} \tag{10}$$

where $\phi \in [0, 2\pi)$. The center of the ellipse within the plane is the point $\mathbf{P} + \alpha_0\mathbf{A} + \beta_0\mathbf{B}$.