Least Squares Fitting of Parallel Lines to Points in 2D

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1 Introduction

A set of n 2-dimensional points $\{\mathbf{P}_i\}_{i=0}^{n-1}$ is believed to live approximately on two parallel lines. The goal is to use a least-squares fitting algorithm to determine the parallel lines.

The motivation for the problem is an application where a directional light illuminates a cylinder, casting a shadow on a plane. A camera captures the projection as a white background with the shadow rendered as a dark rectangular strip (ignoring the cylinder caps). An image processing algorithm determines points on the edges of the shadow. In theory, the two edges are parallel, but noise causes the detected points to lie only approximately on lines.

2 Specialization from Fitting of Cylinder to Points in 3D

The algorithm is the specialization of that used for fitting a cylinder to 3D points; see *Section 7: Fitting a Cylinder to 3D points* in the document Least Squares Fitting of Data by Linear or Quadratic Structures. In that document a cylinder with center $\mathbf{C}$, unit-length axis direction $\mathbf{V}$ and radius $r > 0$ is defined by the quadratic equation

$$(\mathbf{P} - \mathbf{C})^{\mathsf{T}}(I - \mathbf{V}\mathbf{V}^{\mathsf{T}})(\mathbf{P} - \mathbf{C}) - r^2 = 0 \tag{1}$$

where $\mathbf{P}$ is any point on the cylinder. The geometric interpretation of the equation is that the projection of $\mathbf{\Delta} = \mathbf{P} - \mathbf{C}$ onto a plane perpendicular to the cylinder axis has length r. The projection is $\mathbf{Q} = \mathbf{\Delta} - (\mathbf{V} \cdot \mathbf{\Delta})\mathbf{V}$. The squared length is $\mathbf{Q}^{\mathsf{T}}\mathbf{Q}$. Equation (1) is equivalent to $\mathbf{Q}^{\mathsf{T}}\mathbf{Q} = r^2$.

In three dimensions, the infinite cylinder is the set of points equidistant from a line, where the common distance is r and the line is parameterized by $\mathbf{C} + t\mathbf{V}$. In two dimensions, the set of points equidistant from a line consists of two parallel lines, each line a distance r from $\mathbf{C} + t\mathbf{V}$. Equation 1 applies equally well in two dimensions.

Let $\mathbf{V} = (\cos\theta, \sin\theta)$ for $\theta \in [0, 2\pi)$. A unit-length vector perpendicular to $\mathbf{V}$ is $\mathbf{U} = (-\sin\theta, \cos\theta)$. For later use in the derivation, observe that $I - \mathbf{V}\mathbf{V}^{\mathsf{T}} = \mathbf{U}\mathbf{U}^{\mathsf{T}}$, $\mathbf{V}'(\theta) = \mathbf{U}(\theta)$ and $\mathbf{U}'(\theta) = -\mathbf{V}(\theta)$.

If $\mathbf{C}$ is a center for the infinite cylinder, then any point $\mathbf{C} + t\mathbf{V}$ can be chosen as a center. To eliminate this degree of freedom, it is sufficient to choose $\mathbf{C}$ to have no $\mathbf{V}$ component; that is, the center is constrained to satisfy the condition $\mathbf{V} \cdot \mathbf{C} = 0$. In the 2-dimensional setting, the center can be chosen as

$$\mathbf{C} = k\mathbf{U} \tag{2}$$

where k is an unknown quantity to be estimated by the least-squares fitting algorithm.

The sample points are represented by $\mathbf{P}_i = \mu_i\mathbf{U} + \nu_i\mathbf{V}$. It is convenient to subtract out the average of the points. Let $\mathbf{A} = (\sum_{i=0}^{n-1} \mathbf{P}_i)/n$ and replace each $\mathbf{P}_i$ by $\mathbf{P}_i - \mathbf{A}$. The replacement points have the property $\sum_{i=0}^{n-1} \mathbf{P}_i = \mathbf{0}$, which implies

$$\sum_{i=0}^{n-1} \mu_i = 0, \quad \sum_{i=0}^{n-1} \nu_i = 0 \tag{3}$$

using linear independence of $\mathbf{U}$ and $\mathbf{V}$. Observe that $\mu_i = \mathbf{U} \cdot \mathbf{P}_i$ and $\nu_i = \mathbf{V} \cdot \mathbf{P}_i$, so $d\mu_i/d\theta = -\nu_i$ and $d\nu_i/d\theta = \mu_i$.

A least-squares error function is

$$
\begin{aligned}
E(r^2, k, \theta) &= \frac{1}{n} \sum_{i=0}^{n-1} \left((\mathbf{P}_i - \mathbf{C})^\mathsf{T} \left(I - \mathbf{V}\mathbf{V}^\mathsf{T} \right) (\mathbf{P}_i - \mathbf{C}) - r^2 \right)^2 \\
&= \frac{1}{n} \sum_{i=0}^{n-1} \left((\mathbf{P}_i - \mathbf{C})^\mathsf{T} \left(\mathbf{U}\mathbf{U}^\mathsf{T} \right) (\mathbf{P}_i - \mathbf{C}) - r^2 \right)^2 \\
&= \frac{1}{n} \sum_{i=0}^{n-1} \left((\mu_i(\theta) - k)^2 - r^2 \right)^2
\end{aligned} \tag{4}
$$

The goal is to determine the parameters r^2, k and θ that minimize $E(r^2, k, \theta)$. The minimum point is a solution to the system of equations $\nabla E(r^2, k, \theta) = \mathbf{0}$, which consists of three nonlinear equations in three unknowns.

Various summations involving the sample points occur in the derivation. It is convenient to define

$$
S_{pq} = \frac{1}{n} \sum_{i=0}^{n-1} \mu_i^p \nu_i^q \tag{5}
$$

for nonnegative integers p and q. For the points $\mathbf{P}_i$ with zero average, equation (3) implies $S_{10} = 0$ and $S_{01} = 0$.

3 Solving the Gradient System of Equations

The first-order derivative of $E(r^2, k, \theta)$ with respect to r^2 is

$$
\frac{\partial E}{\partial r^2} = \frac{-2}{n} \sum_{i=0}^{n-1} \left((\mu_i - k)^2 - r^2 \right) \tag{6}
$$

The partial derivative is zero when

$$
\frac{1}{n} \sum_{i=0}^{n-1} \left((\mu_i - k)^2 - r^2 \right) = 0 \tag{7}
$$

Solving for r^2 produces

$$
r^2 = \frac{1}{n} \sum_{i=0}^{n-1} (\mu_i - k)^2 = \frac{1}{n} \sum_{i=0}^{n-1} \mu_i^2 - 2k \, \frac{1}{n} \sum_{i=0}^{n-1} \mu_i + k^2 = S_{20} + k^2 \tag{8}
$$

where the last equality uses the condition $S_{10} = 0$.

The first-order derivative of $E(r^2, k, \theta)$ with respect to k is

$$
\frac{\partial E}{\partial k} = \frac{-4}{n} \sum_{i=0}^{n-1} \left((\mu_i - k)^2 - r^2 \right) (\mu_i - k) \tag{9}
$$

The partial derivative is zero when

$$
\begin{aligned}
0 &= \frac{1}{n} \sum_{i=0}^{n-1} \left((\mu_i - k)^2 - r^2 \right) (\mu_i - k) \\
&= \frac{1}{n} \sum_{i=0}^{n-1} \left((\mu_i - k)^2 - r^2 \right) \mu_i - \frac{k}{n} \sum_{i=0}^{n-1} \left((\mu_i - k)^2 - r^2 \right) \\
&= \frac{1}{n} \sum_{i=0}^{n-1} \left((\mu_i - k)^2 - r^2 \right) \mu_i \\
&= \frac{1}{n} \sum_{i=0}^{n-1} \left(\mu_i^3 - 2k\mu_i^2 + (k^2 - r^2)\mu_i \right) \\
&= \frac{1}{n} \sum_{i=0}^{n-1} \mu_i^3 - 2k \frac{1}{n} \sum_{i=0}^{n-1} \mu_i^2
\end{aligned} \tag{10}
$$

where the third equality is true because of equation (7) and the fifth equality is true because $S_{10} = 0$. Solving for k produces

$$k = \frac{1}{2} \frac{\frac{1}{n}\sum_{i=0}^{n-1} \mu_i^3}{\frac{1}{n}\sum_{i=0}^{n-1} \mu_i^2} = \frac{1}{2}\frac{S_{30}}{S_{20}} \tag{11}$$

The first-order derivative of $E(r^2, k, \theta)$ with respect to θ is

$$\frac{\partial E}{\partial \theta} = \frac{-4}{n} \sum_{i=0}^{n-1} \left((\mu_i - k)^2 - r^2\right)(\mu_i - k)\nu_i \tag{12}$$

where $d\mu_i/d\theta = -\nu_i$. The partial derivative is zero when

$$\begin{array}{rcl} 0 & = & \frac{1}{n}\sum_{i=0}^{n-1} \left((\mu_i - k)^2 - r^2\right)(\mu_i - k)\nu_i \\ & = & \frac{1}{n}\sum_{i=0}^{n-1} \left(\mu_i^2 - 2k\mu_i + (k^2 - r^2)\right)(\mu_i\nu_i - k\nu_i) \\ & = & \frac{1}{n}\sum_{i=0}^{n-1} \left(\mu_i^2 - 2k\mu_i - S_{20}\right)(\mu_i\nu_i - k\nu_i) \\ & = & \frac{1}{n}\sum_{i=0}^{n-1} \left(\mu_i^3\nu_i - 2k\mu_i^2\nu_i - S_{20}\mu_i\nu_i - k\mu_i^2\nu_i + 2k^2\mu_i\nu_i + S_{20}k\nu_i\right) \\ & = & \frac{1}{n}\sum_{i=0}^{n-1} \left(\mu_i^3\nu_i - 2k\mu_i^2\nu_i - S_{20}\mu_i\nu_i - k\mu_i^2\nu_i + 2k^2\mu_i\nu_i\right) \\ & = & S_{31} - 3kS_{21} + (2k^2 - S_{20})S_{11} \\ & = & S_{31} - \frac{3}{2}\frac{S_{30}S_{21}}{S_{20}} + \left(\frac{1}{2}\frac{S_{30}^2}{S_{20}^2} - S_{20}\right)S_{11} \end{array} \tag{13}$$

The third equality uses equation (8). The fifth equality uses $S_{01} = 0$. The seventh equality uses equation (11). Define $\sigma = \sin\theta$ and $\gamma = \cos\theta$. Multiply equation (13) by $2S_{20}^2$ to produce

$$F(\sigma, \gamma) = 2S_{31}S_{20}^2 - 3S_{30}S_{21}S_{20} + (S_{30}^2 - 2S_{20}^3)S_{11} = 0 \tag{14}$$

The function F is a polynomial in σ and γ of degree 8. The degree follows from the fact that S_{pq} has degree $p + q$. Define $G(\sigma, \gamma) = \sigma^2 + \gamma^2 - 1$, a polynomial of degree 2 which must also be zero. Observe that the term is generally S_{20} positive for practical data sets. The only way it can be zero is if all μ_i are zero, which implies the samples are $\mathbf{P}_i = \nu_i\mathbf{V}$; that is, the samples are all on a single line.

We now have two polynomial equations, $F(\sigma, \gamma) = 0$ and $G(\sigma, \gamma) = 0$. Elimination theory allows us to reduce this to a single polynomial equation $H(\sigma) = 0$ whose degree is no larger than 16, the product of the degrees of F and G. The roots of the polynomial are candidates for minimizing $E(r^2, k, \theta)$. During the elimination, an equation will be constructed that defines γ in terms of σ, so it is not necessary to attempt using $\gamma = \pm\sqrt{1 - \sigma^2}$ when evaluating E at the candidates.

The least-squares error $E(r^2, k, \theta)$ in equation (4) can be written by expansion,

$$\begin{array}{rcl} E & = & \frac{1}{n}\sum_{i=0}^{n-1} \left((\mu_i - k)^2 - r^2\right)^2 \\ & = & \frac{1}{n}\sum_{i=0}^{n-1} \left(\mu_i^2 - 2k\mu_i + k^2 - r^2\right)^2 \\ & = & \frac{1}{n}\sum_{i=0}^{n-1} \left(\mu_i^2 - 2k\mu_i - S_{20}\right)^2 \\ & = & \frac{1}{n}\sum_{i=0}^{n-1} \left(\mu_i^4 + 4k^2\mu_i^2 + S_{20}^2 - 4k\mu_i^3 - 2S_{20}\mu_i^2 + 4kS_{20}\mu_i\right) \\ & = & S_{40} - 4kS_{30} + 4k^2S_{20} - S_{20}^2 \end{array} \tag{15}$$

where the third equality uses equation (8) and the fifth equality uses $S_{10} = 0$.

4 Preprocessing the Polynomial $F(\sigma, \gamma)$

Evaluation of the polynomial involves summations over products of components of $\mathbf{P}_i = (x_i, y_i)$. It is convenient to define

$$Z_{pq} = \frac{1}{n} \sum_{i=0}^{n-1} x_i^p y_i^q \tag{16}$$

for nonnegative integers p and q. For the points $\mathbf{P}_i$ with zero average, it is clear that $Z_{10} = 0$ and $Z_{01} = 0$. The terms S_{pq} can be converted to expressions involving σ, γ and Z_{pq},

$$\begin{array}{rcl} S_{11} & = & -\sigma\gamma Z_{20} + (\gamma^2 - \sigma^2) Z_{11} + \sigma\gamma Z_{02} \\ S_{20} & = & \sigma^2 Z_{20} - 2\sigma\gamma Z_{11} + \gamma^2 Z_{02} \\ S_{30} & = & -\sigma^3 Z_{30} + 3\sigma^2\gamma Z_{21} - 3\sigma\gamma^2 Z_{12} + \gamma^3 Z_{03} \\ S_{21} & = & \sigma^2\gamma Z_{30} + \sigma(\sigma^2 - 2\gamma^2) Z_{21} + \gamma(\gamma^2 - 2\sigma^2) Z_{12} + \sigma\gamma^2 Z_{03} \\ S_{40} & = & \sigma^4 Z_{40} - 4\sigma^3\gamma Z_{31} + 6\sigma^2\gamma^2 Z_{22} - 4\sigma\gamma^3 Z_{13} + \gamma^4 Z_{04} \\ S_{31} & = & -\sigma^3\gamma Z_{40} + \sigma^2(3\gamma^2 - \sigma^2) Z_{31} + 3\sigma\gamma(\sigma^2 - \gamma^2) Z_{22} + \gamma^2(\gamma^2 - 3\sigma^2) Z_{13} + \sigma\gamma^3 Z_{04} \end{array} \tag{17}$$

The terms Z_{pq} can be precomputed and the root finder can use the expressions in equation (17) during evaluations of $F(\sigma, \gamma)$.

Rather than using a generic approach to elimination, say, using the Sylvester or Bézout determinant, the preprocessing can be performed as shown next. Let $S_{pq} = a_{pq}(\sigma) + \gamma b_{pq}(\sigma)$, obtained by replacing γ^2 by $1 - \sigma^2$ in the expressions as often as is necessary to reduce the expression to the desired form. The expressions are

$$\begin{array}{rcl} S_{11} & = & \left[(1 - 2\sigma^2) Z_{11}\right] + \gamma \left[\sigma(Z_{02} - Z_{20})\right] \\ S_{20} & = & \left[\sigma^2 Z_{20} + (1 - \sigma^2) Z_{02}\right] + \gamma \left[-2\sigma Z_{11}\right] \\ S_{30} & = & \left[-\sigma^3 Z_{30} - 3\sigma(1 - \sigma^2) Z_{12}\right] + \gamma \left[3\sigma^2 Z_{21} + (1 - \sigma^2) Z_{03}\right] \\ S_{21} & = & \left[\sigma(\sigma^2 - 2(1 - \sigma^2)) Z_{21} + \sigma(1 - \sigma^2) Z_{03}\right] + \gamma \left[\sigma^2 Z_{30} + ((1 - \sigma^2) - 2\sigma^2) Z_{12}\right] \\ S_{40} & = & \left[\sigma^4 Z_{40} + 6\sigma^2(1 - \sigma^2) Z_{22} + (1 - \sigma^2)^2 Z_{04}\right] + \gamma \left[-4\sigma^3 Z_{31} - 4\sigma(1 - \sigma^2) Z_{13}\right] \\ S_{31} & = & \left[\sigma^2(3(1 - \sigma^2) - \sigma^2) Z_{31} + (1 - \sigma^2)((1 - \sigma^2) - 3\sigma^2) Z_{13}\right] \\ & & + \gamma \left[-\sigma^3 Z_{40} + 3\sigma(\sigma^2 - (1 - \sigma^2)) Z_{22} + \sigma(1 - \sigma^2) Z_{04}\right] \end{array} \tag{18}$$

An important observation is that each σ-polynomial is either an even function, where only even powers of σ occur, or an odd function, where only odd powers of σ occur. This leads to

$$F(\sigma, \gamma) = f_0(\sigma) + \gamma f_1(\sigma) \tag{19}$$

where $f_0(\sigma)$ is an even polynomial of degree 8 and $f_1(\sigma)$ is an odd polynomial of degree 7. This implies $f_0(\sigma)$ is a polynomial of degree 4 in the variable σ^2 and $f_1(\sigma)/\sigma$ is a polynomial of degree 3 in the variable σ^2. Another consequence of the even or odd polynomials is that $F(-\sigma, -\gamma) = F(\sigma, \gamma)$. This is consistent with the geometry of the problem. If $\mathbf{V} = (\gamma, \sigma)$ is a direction vector for the parallel lines, then so is $-\mathbf{V} = (-\gamma, -\sigma)$.

If $f_1(\sigma)$ is not identically zero, solve $F = 0$ for $\gamma = -f_0/f_1$, and then substitute into $\sigma^2 + \gamma^2 - 1 = 0$. The polynomial equation is $H(\sigma) = \sigma^2 f_1^2 + (f_0^2 - f_1^2) = 0$, which is degree 16 in σ. The polynomial H is an even function, so it is degree 8 in the variable σ^2. Compute the nonnegative roots σ and corresponding $\gamma = -f_0(\sigma)/f_1(\sigma)$ values and then locate the minimizer. If $f_1(\sigma)$ is identically zero, then we need only solve $f_0(\sigma) = 0$ and then try pairs $(\sigma, \pm\sqrt{1-\sigma^2})$ to locate the minimizer.

5 Returning the Results

The output of the minimization for $E(r^2, k, \theta)$ is $\mathbf{V} = (\cos\theta, \sin\theta)$, r^2 and $\mathbf{C}$. Internally, the center point is $\mathbf{C}' = \mathbf{A} + k\mathbf{U}$, where k is determined by the minimization, $\mathbf{A}$ is the average of the input points and $\mathbf{U} = (-\sin\theta, \cos\theta)$. For consistency, the returned center has the property $\mathbf{V} \cdot \mathbf{C} = 0$; that is, $\mathbf{C} = \mathbf{C}' - (\mathbf{V} \cdot \mathbf{C}')\mathbf{V}$.

The source code is found in the file ApprParallelLines2.h.