Documentation Updates for Geometric Tools

July 5, 2022. Triangulation by Ear Clipping
Fixed a typographical error in Figure 2. I had "edge list" when it should have been "ear list."

June 29, 2022. Principal Curvatures of Surfaces
This is a much need revision of an old document. The previous version had two sections on principal curvatures for implicit surfaces. The second of those was not relevant to the goal of the document. The first of those generated a quadratic form whose extreme eigenvalues are the principal curvatures. However, I lifted the problem back to the original dimension

$$-\left(I - \frac{\nabla F}{|\nabla F|}\frac{\nabla F^{\mathsf{T}}}{|\nabla F|}\right)\frac{D^2F}{|\nabla F|}\,\boldsymbol{v} = \kappa\boldsymbol{v}$$

The problem with lifting before solving an eigensystem is that the lifted system has a zero-valued eigenvalue (because of the lifting) but if one of the principal curvatures is 0, you have to decide which eigenvector to use for the principal direction. It is better to solve the eigensystem in the reduced dimension and then lift the eigenvectors to the original dimension. The current version of the document does this. The modifications also include more details about the quadratic forms and restricted quadratic forms. And I modified the curvature signs to be consistent whether an implicitly defined surface or parametrically defined surface.

June 20, 2022. Intersection of Convex Objects: The Method of Separating Axes
I discovered that the PDF is out of sync with the LaTeX source. The document was significantly modified on 14 February 2019. The documentation history shows that on that day, a group of PDF files were modified to update hyperlinks to the latest GTE source code. It appears I failed to modify the update history and to post the new document.

June 8, 2022. Natural Spline Interpolation
This is a major rewrite of the document. The previous version discussed only natural cubic splines. The method of solution for free splines and clamped splines was based on the classical approach—solving a tridiagonal linear system which requires $O(n)$ time. The closed splines were presented in terms of a matrix system that can be reduced quickly to a block upper-triangular system, but the details were sparse and not implemented in the GTE class NaturalSplineCurve. The new document provides more details about the block upper-triangular approach and shows that it is $O(n)$ and applies whether the spline is free, clamped or closed. Moreover, the concept applies to higher-degree splines. Some symbolic algebra allows an implementation that is extremely efficient in memory and CPU time. Examples are presented in the document. New classes were added to GTE, namely, NaturalCubicSpline and NaturalQuinticSpline.

May 26, 2022. Alhazen's Problem: Reflection Point on a Sphere
This is a revision that includes references to the trigonometric formulation of the problem and to the quartic-polynomial formulation. I added pseudocode for computing the reflection point for both formulations of the problem.

March 4, 2022. Triangulation by Ear Clipping
Section 4 of the document was modified to choose the reflex vertex R inside the triangle $\langle M, I, P\rangle$ that is closest to M. This is simpler to compute than by choosing R that minimizes the angle between $\langle M, I\rangle$ and $\langle M, R\rangle$. Thanks to Mark Spoor for providing the idea.

February 25, 2022. Fit a Cone to Ellipse and Points
A new document that describes how to fit a cone to known elliptical cross sections and some additional points. If the cross sections are provided as point samples approximately on the ellipses, the ellipse points can be extracted and each ellipse fitted with a quadratic function. GTE has code for this, ApprCone3EllipseAndPoints.h, and a sample mathematics application FitConeByEllipseAndPoints.

January 26, 2022. Computing Impulsive Forces for Colliding Contact
The document is new and describes a variation on computing impulsive forces for colliding contact. The algorithm is an extension of the one discussed in my book *Game Engine Design, 2nd edition* in Section 6.2.2.

December 21, 2021. Intersection of a Box and a Finite Cylinder
This is a new document that describes several algorithms for the test-intersection query between a box and a finite cylinder. The GTE code already existed for the query, but it used an LCP approach which turned out to work correctly when using exact arithmetic but had significant rounding errors when using floating-point arithmetic. The code has been modified using the third algorithm described in the PDF.

December 1, 2021. Intersection of a Cylinder and a Plane
This is a major rewrite of the previous version. The mathematical derivation included constructing the major axis of the ellipse of intersection for an infinite cylinder and a plane. The axis direction was $\boldsymbol{A} = (I - \boldsymbol{N}\boldsymbol{N}^{\mathsf{T}})\boldsymbol{W}/|(I - \boldsymbol{N}\boldsymbol{N}^{\mathsf{T}})\boldsymbol{W}|$ where $\boldsymbol{N}$ is the plane normal and $\boldsymbol{W}$ is the cylinder axis direction. This formula has a singularity when $\boldsymbol{N}$ and $\boldsymbol{W}$ are parallel. The corresponding Geometric Tools source code (IntrPlane3Cylinder.h) failed in this case. A different construction for $\boldsymbol{A}$ is now used. The old source code only supported infinite cylinders, but the new source code supports infinite and finite cylinders. The PDF describes the algorithms for both.

October 31, 2021. Perspective Projection of an Ellipse onto a Plane in 3D
This is a major rewrite of PerspectiveProjectionEllipse.pdf that describes perspective projections of a 3D ellipse to a projection plane. When the eye point is not in the plane of the ellipse, the projection set can be an ellipse, a parabola or a branch of a hyperbola. When the eye point is in the plane of the ellipse, the projection set is a line, ray or segment, as described in Perspective Projection of an Ellipse onto a Line in 2D.

October 29, 2021. Perspective Projection of an Ellipse onto a Line in 2D
This is a new document that will be used to motivate a rewrite of Perspective Projection of an Ellipse that was written to show how to project a 3D ellipse perspectively onto a projection plane. The 3D document assumes that all 3D ellipse points are projectable onto the plane, leading to a 2D ellipse of projection. However, if the eye point is in the plane of the 3D ellipse, the problem reduces to the perspective projection of an ellipse onto a line in 2D. Also, if the 3D ellipse points are not all projectable, the projection can be a parabola or a hyperbola. I will be rewriting the 3D document shortly, renaming it to *Perspective Projection of an Ellipse onto a Plane in 3D*.

September 19, 2021. Distance to Circles in 3D
The source code did not match the description for computing the distance robustly. It computed a closest point first, which was not robust, and then computed the distance using the input point and that closest point. The robustness problem occurs when the input point is nearly on the normal line $\boldsymbol{C} + t\boldsymbol{N}$, where $\boldsymbol{C}$ is the circle center and $\boldsymbol{N}$ is a unit-length normal to the plane of the circle. The distance computation now matches the PDF and is robust. The closest-point code was revised to be robust, and the PDF was expanded to describe the new code.

September 7, 2021. Least Squares Fitting of Data by Linear or Quadratic Structures
Fixed a bug in ApprCone3 in the block of code shown at the end of Listing 19 of the PDF. The roots were

supposed to be searched for the minimum, but the code for updating minItem did not update the minError. The quartic roots are generally stored in increasing order except when a repeated root 0 is found (for the depressed quartic). The cone fitting should not lead to a root of 0, so the previous code worked by luck.

September 5, 2021. Intersection of Cylinders
This is a significant rewrite of the previous version of the PDF based on trying to implement the algorithm in GTE. The minimization phase of the algorithm was more challenging to implement than the previous PDF suggested.

August 24, 2021. A Robust Eigensolver for 3×3 Symmetric Matrices
Added more details to the document and revised the content so that the PDF and the source code match.

May 28, 2021. Least Squares Fitting of Data by Linear or Quadratic Structures
Modified the document to include a section that describes fitting a cylinder to a triangle mesh.

May 22, 2021. Distance from an Oriented Box to a Cone Frustum
This is a new document describing how to compute the distance between an oriented box and a cone frustum. Source code has been added to the Geometric Tools distribution, including a sample application to illustrate the algorithm.

April 30, 2021. Intersection of a Triangle and a Cylinder
Added more details about the triangle-cylinder test-intersection query. Replaced the references to Wild Magic 5 code with references to the GTEngine code.

October 10, 2020. Least Squares Fitting of Data by Linear or Quadratic Structures
Rewrote the section on fitting hyperellipsoids to points. The new algorithm uses a 2-step gradient descent. The implementations for ApprEllipse2 and ApprEllipsoid3 were modified to use the new algorithm.

September 11, 2020.
Akima Interpolation for Nonuniform 1D Data
GTE: Arbitrary Precision Arithmetic
The Area of Intersecting Ellipses
Least-Squares Fitting of Data with B-Spline Curves
B-Spline Interpolation on Lattices
Clip a Convex Polygon by a Hyperplane
Converting Between Coordinate Systems
Convex Quadratic Programming
Robust Computation of Distance Between Line Segments
Distance Between Point and Triangle in 3D
Distance from a Point to an Ellipse, an Ellipsoid, or a Hyperellipsoid
Distance to Circles in 3D
Intersection of a Box and a Cone or Cone Frustum
Intersection of a Line and a Box
Intersection of a Line and a Cone
Intersection of Moving Circle and Rectangle
Intersection of Moving Sphere and Box
Intersection of Moving Sphere and Triangle
Intersection of Ellipses
Intersection of a Sphere and a Cone
Least Squares Fitting of Data by Linear or Quadratic Structures
Constructing a Cycle Basis for a Planar Graph

Minimum-Area Rectangle Containing a Set of Points
Minimum-Volume Box Containing a Set of Points
Numerical Integration
Representing a Circle or a Sphere with NURBS
A Robust Eigensolver for 2×2 Symmetric Matrices
A Robust Eigensolver for 3×3 Symmetric Matrices
Fitting 3D Data with a Torus
Replaced all the hyperlinks to GTEngine files with hyperlinks to GTE files.

September 11, 2020. Intersection of Ellipses
Sections 4.2 and 4.3 had several places where the case $d_0 = d_1$ was mentioned. The function was incorrectly stated as $f(s) = 2d_0k_0^2/(d_0s - 1)^2 - 1$ but should have been $f(s) = d_0(k_0^2 + k_1^2)/(d_0s - 1)^2 - 1$. Also in Section 4.3, 5th paragraph, there was a term $\sqrt{f(0)}$ that should have been $\sqrt{1 + f(0)}$. Finally, I modified the hyperlinks to point to source code in GTE (version 4) rather than the obsolete GTEngine (version 3).

September 5, 2020. Minimum Volume Box Containing a Set of Points
This is a complete rewrite of the previous version. I had incorrectly stated the conditions for a minimum-volume box. The correct algorithm is described in the new version. The corresponding source code and sample application have been updated accordingly.

August 14, 2020. Thin-Plate Splines
The next-to-last sentence had an equation for the minimum of the functional that had an extra λ term. I removed that term.

August 12, 2020. Approximations to Rotation Matrices and Their Derivatives
This is a new document about approximating a rotation matrix and its derivatives using minimax polynomial approximations. The information in *Robust and Error-Free Geometric Computing* about floating-point issues with the related function evaluation has been greatly expanded in this PDF.

April 28, 2020. Euler Angle Formulas
Fixed some typographical errors in the factorization as $R_{z_0}R_yR_{z_1}$ of Section 2.12.

April 7, 2020. Distance Between Point and Triangle in 3D
Fixed some typographical errors in the comments. Listings 5, 7 and 8 had occurrences of $F(s) = at + d$ but should be $F(s) = as + d$. Line 5 of the pseudocode of Listing 8 had $t > 0$ that should be $s > 0$.

February 21, 2020. Globally C^s and Locally Controllable Interpolation on n-Dimensional Lattices
This is a new document, so to speak. It has been sitting in my to-be-processed queue for 7 years. I thought it was time to post it. The document shows how to generate smooth interpolation on lattice data with C^s continuity.

February 1, 2020. Derivative Approximation by Finite Differences
This is a major revision of the document (from approximately 20 years ago). A clarification was made for centered-difference approximations when the derivative order is even and the error order is odd. Tables of derivative estimates were added, and the previous verbosity of writing down all those linear systems is gone.

December 27, 2019. Least Squares Fitting of Parallel Lines to Points in 2D
This is a new document describing how to fit two clusters of planar points with two parallel lines. This is a specialization of the least squares cylinder fitting in 3D.

December 10, 2019. A Robust Eigensolver for 3x3 Symmetric Matrices
The paragraphs after Listing 2 had the incorrect statement $\boldsymbol{X} = J\boldsymbol{E}$. The correct statement is $\boldsymbol{E} = J\boldsymbol{X}$,

where $\boldsymbol{X}$ is the 2×1 vector whose rows are x_0 and x_1.

December 6, 2019. Rotation Representations
Removed the section on performance that was measured by operation counts. Such counting is nearly always not relevant on current CPUs.

September 10, 2019. Low-Degree Polynomial Roots
Fixed a hyperlink to the Wikipedia page on quartic roots. Minor fixes to pseudocode for quartic root finding. Minor edits of the text.

August 29, 2019. GTEngine: Arbitrary Precision Arithmetic
Fixed various typographical errors. Removed the section on exact computation when square roots are involved (the quadratic field discussion). This section had several problems in the presentation, and the source code is on longer that of GTEngine 3 and 4 code. The material will be presented in expanded form in the forthcoming book *Robust and Error-Free Geometric Computing*.

July 29, 2019. Thin-Plate Splines
Equation (24) had used implicitly the fact that the surface integral is independent of $\boldsymbol{s}$ and replaced $\boldsymbol{s}$ by $\boldsymbol{0}$, even though the comment remained "setting $r = |\boldsymbol{x} - \boldsymbol{s}|$." Consequently, equation (26) contained $\boldsymbol{x}$ rather than $\boldsymbol{x} - \boldsymbol{s}$ and equation (27) had integration region $S(\boldsymbol{0}, \varepsilon)$. Later in the document the balls need to be centered at sample points $\boldsymbol{x}_i$, so using the independence from $\boldsymbol{s}$ in the construction is confusing. I modified these equations to use the center $\boldsymbol{s}$.

May 28, 2019. Intersection of a Box and a Cone or Cone Frustum
Page 5 had an expressions $\boldsymbol{P} - \boldsymbol{U}$ that should have been $\boldsymbol{P} - \boldsymbol{V}$.

April 10, 2019. Intersection of Infinite Cylinders
Fixed various typographical errors involving hat or tilde symbols.

April 2, 2019. Moving Along a Curve with Specified Speed
Fixed various errors in the section involving piecewise-defined curves. The pseudocode also had several errors based on an apparent assumption that the curve segment evaluators were accepting times relative to the minimum time of the segment when in fact they needed to be global times.

April 1, 2019. Perspective Mappings
Added a section that describes how to compute the perspective mapping between two convex quadrilaterals using a homography.

March 3, 2019. Intersection of a Box and a Cone or Cone Frustum
This is an expansion of the original document that provided a test-intersection query between a box and an infinite cone. The modified document handles infinite cones, infinite truncated cones, finite cones and cone frusta. A sample application was added to GTEngine to demonstrate the code.

February 14, 2019.
Converting Between Coordinate Systems
Least Squares Fitting of Data by Linear or Quadratic Structures
Mesh Differential Geometry
Minimum Area Rectangle Containing a Set of Points
Minimum Volume Box Containing a Set of Points
Distance from a Point to an Ellipse, an Ellipsoid, or a Hyperellipsoid
Distance to Circles in 3D
An Approximation for the Inverse Square Root Function

Akima Interpolation for Nonuniform 1D Data
Intersection of a Box and a Cone or Cone Frustum
Intersection of Box and Ellipsoid
Intersection of Ellipses
Reconstructing an Ellipsoid from its Perspective Projection onto a Plane
Fixed broken hyperlinks and updated http to https.

February 13, 2019. Clip a Convex Polygon by a Hyperplane
This is a new document describing test-intersection and find-intersection queries between a convex polygon and a hyperplane in n dimensions. The find-intersection query also produces the clipped polygons when the polygon is split by the hyperplane. The algorithm is designed to be robust when using floating-point arithmetic.

January 21, 2019. Intersection of Moving Sphere and Triangle
This is a complete rewrite of the document. The previous version used Voronoi regions to decompose the path of the sphere into sections that can be searched in-order to test only those features of the triangle that might have the contact point. The document was complicated, and the Wild Magic 5 implementation was buggy. The new version is based on Minkowski sums, shrinking the sphere to a point (its center) and expanding the triangle to a sphere-swept volume. This is the same approach used for computing the intersection of a moving sphere and cone, aligned box and oriented box that is describe in other documents at the website. New files were added to GTEngine to support the query, and there are versions for floating-point arithmetic and for exact arithmetic. A sample mathematics application called MovingSphereTriangle has been added to GTEngine to illustrate the query.

January 16, 2019. Representing a Circle or a Sphere with NURBS
Added the general solution for the quarter circle of degree 4, computing w_1, x_2 and y_1 in terms of w_1.

November 29, 2018. Intersection of a Box and a Cone or Cone Frustum
The previous title of the document is *Intersection of an Oriented Box and a Cone*. That document described a test-intersection query for an oriented box and an infinite cone. The ideas can be extended to apply to aligned boxes and cone frusta. The document was modified to describe the extension. Both the intersection queries in the GTEngine code and the sample test application were modified.

November 28, 2018. Intersection of a Line and a Box
This is a new document that describes the test-intersection and find-intersection queries between a linear component (line, ray, segment) and a box (aligned, oriented). Some of this material is in *3D Game Engine Design, 2nd edition*, but the new document includes many more details and ties the documentation to the GTEngine online source code.

October 30, 2018. Least Squares Fitting of Data by Linear or Quadratic Structures
Expanded the discussion on choosing an initial guess for cone fitting. An implementation of the algorithm is now in the GTEngine code and there is a sample application (FitCone) that illustrates the code.

October 28, 2018. Representing a Circle or a Sphere with NURBS
Fixed several typographical errors involving control points. Added more details about the Bernstein polynomials for the triangle patch that represents an eighth sphere. These details are for derivative computations in newly added classes in GTEngine, NURBSCircle and NURBSSphere, that encapsulate the details of curve and surface construction. Two corresponding sample applications were added to GTEngine. Screen captures are provided in the PDF.

October 9, 2018. A Robust Eigensolver for 3x3 Symmetric Matrices

The paragraph after equation (9) in Section 5.2 had "...are the rows of the matrix $B - \beta_0 I$." The matrix should be $A - \alpha_0 I$. The code listing of Section 6 for the noniterative algorithm has code that computes the eigenvalues eval[0], eval[1] and eval[2]. The code after that had a bug that I fixed in the online source code but failed to update the PDF. The block starting with comment The index i0 corresponds ... should be deleted. The if-else statement in the block after that, which has comment Compute the eigenvectors., has been updated. Finally, I updated the comments in the main block (online code and code in PDF) of the solver to use variable names consistent with those of the PDF. More comments were included to explain the code (which was somewhat cryptic).

October 8, 2018. Least Squares Fitting of Data by Linear or Quadratic Structures
Added a subsection to the section on fitting a cone to points that shows how to choose the initial cone vertex and cone axis direction for the iterative nonlinear least-squares algorithm.

October 3, 2018. Intersection of Moving Sphere and Box
This is a new document that describes how to compute the first time of contact and point of contact between a sphere and a box (considered as solids), each moving with constant linear velocity. An implementation is available at the website and a sample application was added to demonstrate (GeometricTools/GTEngine/Samples/Mathematics/MovingSphereBox).

October 3, 2018. Intersection of Moving Circle and Rectangle
Minor changes to make the presentation comparable with 3D extension of moving sphere-box intersections.

September 28, 2018. Distance from a Point to an Ellipse, an Ellipsoid, or a Hyperellipsoid
The function GetRoot in the pseudocode of Listing 4 had the initialization s0 = z1 - 1 that should have been s0 = z2 - 1. This appears to have been a cut-and-paste error from Listing 2. The actual source code in GteDistPointHyperellipsoid.h has the correct initialization.

September 18, 2018. Intersection of Moving Circle and Rectangle
The pseudocode for InRegion4 had variable result that was intended to be the pair contactTime and contactPoint.

September 16, 2018. Least Squares Fitting of Data by Linear or Quadratic Structures
This is a major revision of the document. Some new algorithms were added for fitting circles, spheres, ellipses, ellipsoids, and cones. The descriptions have been expanded and pseudocode has been included.

September 16, 2018. Fitting 3D Data with a Torus
This is a new document describing several algorithms for fitting 3D points with a torus. The source code implementation already exists in the current GTEngine distribution.

September 11, 2018. A Fast and Accurate Estimate for SLERP
This is a new document that is a revision of a paper I published in 2011 in the Journal of Graphics, GPU, and Game Tools (later called simply the Journal of Graphics Tools). The error analysis of the paper was incorrect. The new document fixes that, using the correct error-balancing approach I had intended in the original paper.

August 10, 2018. Least Squares Fitting of Segments by Line or Plane
This is a new document that describes a simple extension of least-squares fitting of points by line or plane to least-squares fitting of line segments by line or plane.

August 10, 2018. Least Squares Fitting of Data
Fixed some typographical errors.

August 1, 2018. Intersection of Moving Circle and Rectangle

Added a description of how to reduce the amount of code by sharing based on geometric symmetry.

July 31, 2018. Intersection of Moving Circle and Rectangle
This is a new document that describes how to compute the first time of contact and point of contact between a circle and a rectangle (considered as solids), each moving with constant linear velocity. An implementation is available at the website and a sample application was added to demonstrate (GeometricTools/GTEngine/Samples/Mathematics/MovingCircleRectangle).

June 8, 2018. Numerical Integration
Fixed the typographical error in equation (2), replacing 1/2 by 1/4. Added equation numbers to the displayed equations.

June 7, 2018. B-Spline Interpolation on Lattices
This is a major rewrite of the original document that was written as part of the draft for my *Ridges in Image and Data Analysis* book. The source code implementation in Wild Magic 5 is also quite old. The GTEngine source code is a complete rewrite with a sample application to illustrate the interpolators. That application is used as a unit test for the code.

June 5, 2018. Distance from Line to Rectangle in 3D
Fixed three typographical errors.

May 2, 2018. Robust Computation of Distance Between Line Segments
Fixed a typographical error in the pseudocode for region 8.

April 28, 2018. Distance from Line to Rectangle in 3D
Fixed two typographical errors.

April 22, 2018. Distance from Line to Rectangle in 3D
Added a section for computing the distance between a rectangle and either a ray or a segment.

April 7, 2018. C^1 Quadratic Interpolation of Meshes
This document is a major rewrite of the previous version. That version was missing a lot of mathematical details that are important to understand the algorithm. I replaced the section about interpolation for general meshes with information about how to generate surface parameters (texture coordinates) for a general mesh and then apply the Cendes-Wong algorithm to each positional component of the mesh.

March 31, 2018. Distance from Line to Rectangle in 3D
This is a new document that describes a robust and efficient algorithm for computing the distance between a line and a rectangle in 3D, including computing a pair of closest points—one on the line and one on the rectangle (possibly the same point if the line intersects the rectangle). The document is intended to replace the discussion in Section 10.9.2 of *Geometric Tools for Computer Graphics*, a section that is full of annyoing errors. See the book corrections page for details about the errors.

January 21, 2017. Thin-Plate Splines
The constants for the Green's function $G(r)$ of equation (28) were listed as a_0 and b_1 but the text after the equation mentioned a_0 and a_1. The b_1 needed to be a_1, but there might still be some confusion because I used a_0 and a_1 in the coefficients for the function for thin-plate splines without smoothing. I modified equation (28) to use α as the constant and specified the values based on dimension n. I had also made statements about absorbing the constant (now α) into the linear solver for the thin-plate splines—with or without smoothing. Without smoothing, this is not a problem. With smoothing, the solution as stated was not correct; it needed a division of λ by α. To avoid this, I removed the comments about absorbing the constants. I also added some more details about the construction of the linear system for thin-plate splines

with smoothing.

December 12, 2017. Least Squares Fitting of Data
A term in equation (17) was missing a transpose operator.

December 10, 2017. Convex Quadratic Programming
Added a new document that shows how to solve convex quadratic programming problems formulated as linear complementarity problems (LCPs). Many standard geometric queries can be formulated as convex quadratic programming problems, including distance and closest-point queries and including intersection testing. When using floating-point arithmetic, LCPs can suffer from rounding errors—as can any attempt to solve geometric queries when the algorithm requires cases to handle non-parallel or parallel situations. The new document discusses how to use rational arithmetic in order to obtain an exact solution. When normalized vectors are part of the algorithm, one can avoid the rounding and approximationg errors introduced during normalization by using real quadratic fields.

December 9, 2017. GTEngine: Arbitrary Precision Arithmetic
Fixed some typographical errors. Added new sections to discuss how to obtain exact results when square roots are involved, although the output is produced by a mixture of rational arithmetic and symbolic manipulation. The exact results use real quadratic fields, and the approach is useful when solving convex quadratic programming problems as linear complementarity programs (such as computing the distance between standard geometric solids).

October 29, 2017. Distance Between Point and Triangle in 3D
Added code listings for all 7 regions. Fixed some inconsistencies with variable names in the code listings. Polished some of the discussion and added website links to the actual C++ source code.

October 15, 2017. Interpolation of Rigid Motions in 3D
Added a new document that describes how to compute the geodesic path connecting two rigid transformations involving rotations and translations.

October 11, 2017. In all the PDF documentation, replaced the copyright notices and *All Rights Reserved* by the Creative Commons Attribution 4.0 International License.

April 16, 2017. Intersection of Cylinders
Modified the pseudocode to include all the parameters in the calls to the functions F(t,*) and FDer(t,*).

April 2, 2017. GTEngine: Arbitrary Precision Arithmetic
Fixed the title (wrong title due to a cut-and-paste error). Modified the paragraph that mentioned overflow occurs for values in the interval $[0, 2^{-149})$, which should be fully open $(0, 2^{-149})$.

November 26, 2016. The Area of Intersecting Ellipses
Fixed broken hyperlinks.

November 25, 2016. Distance from a Point to an Ellipse, an Ellipsoid, or a Hyperellipsoid
Fixed a typographical error in the pseudocode of Listing 4. The distance term in the block of code where y0, y1, and y2 are positive with g not equal to zero was missing the squared x2-y2 term.

November 20, 2016. Least Squares Fitting of Data
Fixed several typographical errors and added equation numbers. New sections were added for fitting circles and spheres to points by estimating coefficients for quadratic equations. New section for fitting a k-flat to points in n dimensions. Fixed broken links (because of a problem with spaces in arguments to the `\href` command). Added new links to source code implementations of the algorithms.

September 21, 2016. Derivative Approximation by Finite Differences
Fixed a couple of typographical errors and added clarification about how equation (17) is derived.

September 20, 2016. A Robust Eigensolver for 3x3 Symmetric Matrices
Added the discussion about a noniterative eigensolver for 3×3 real-valued symmetric matrics.

September 9, 2016. Distance from a Point to an Ellipse, an Ellipsoid, or a Hyperellipsoid
Fixed some typographical errors in pseudocode. The terms ey0 and ey1 were replaced by the correct terms numer0 and numer1.

July 23, 2016. A Robust Eigensolver for 2x2 Symmetric Matrices
A new document that describes the robust algorithm for computing the eigenvalues and eigenvectors of a 2x2 symmetric matrix. The source code is in GteSymmetricEigensolver2x2.h but is also listed in this document.

May 31, 2016. Akima Interpolation for Nonuniform 1D Data
The pseudocode listed header files for GTEngine, but the body of main had code style and references to Wild Magic 5 code. Also, the file GteIntpAkimaNonuniform1.h is referenced in the pseudocode but was missing from the GTEngine distribution. It is now in the distribution.

May 31, 2016. Mesh Differential Geometry
This is a major revision of the document. The new version shows how to estimate vertex tangents and vertex normals for a parameterized mesh (or mesh with texture coordinates) at each vertex using the entire set of triangles that share the vertex. The algorithm is useful for constructing the vertex normals and tangents whose interpolated values are used in a pixel shader for tangent-space normal mapping. An implementation of these algorithms will be posted with the update GTEngine version 2.6. This construction may be used as a replacement for the one-triangle (biased) estimate in the now obsolete document *Estimating a Tangent Vector for Bump Mapping (*BumpMapping.pdf*)*. The bump mapping document has been removed, but the link to it is now redirected to the new version of this document.

April 24, 2016. Approximating an Ellipse by Circular Arcs
Removed the references to Wild Magic 4 code and added pseudocode for the actual fitting algorithm. After posting the revision, a minor change was added in the pseudocode (changed an occurrence of circles[] to centers[]).

April 7, 2016. Mesh Differential Geometry
Fixed a typographical error in the $\boldsymbol{T}(s)$ equation on page 4 [$\theta(t)$ needed to be $\theta(s)$].

April 2, 2016. Constructing a Cycle Basis for a Planar Graph
At the request of a reader, the document was rewritten again, this time eliminating the discussion of the min-heap speed up that was interleaved in the algorithm. The discussion is easier to follow and it is clearer now how the nesting of cycles is handled. More pseudocode was added. The GTEngine source code (GteMinimalCycleBasis.h) was rewritten to match the document. Without the min-heap support, the performance is still reasonable. Running on an Intel i7-4790 CPU (3.6 GHz), a graph with 32K vertices and 32K edges required 12 seconds to complete, including the validation that the graph is planar. And this run used exact rational arithmetic.

March 14, 2016. Converting Between Coordinate Systems
Fixed a broken link to source code and a minor typographical error.

March 7, 2016. Constructing a Cycle Basis for a Planar Graph
This is a complete rewrite of the document *The Minimal Cycle Basis for a Planar Graph*. The PDF name is the same but the title is new (using correct terminology). The Wild Magic 5 sample application was ported,

but testing showed that the implementation was incorrect. The GTEngine implementation has been tested more heavily, and code was added to test whether the input graph is planar (requires exact arithmetic).

December 3, 2015. Distance from a Point to an Ellipse, an Ellipsoid, or a Hyperellipsoid
Fixed broken links to source code.

November 1, 2015. Thin Plate Splines
Provided more details for the construction of the Green's functions. Fixed some typographical errors and incorrect equation references. Added comments about the unboundedness of the Green's functions for dimensions 4 and larger.

September 20, 2015. Intersection of Rectangle and Ellipse
Fixed the pseudocode regarding intersections and overlaps.

September 20, 2015. Intersection of Box and Ellipse
Fixed the pseudocode regarding intersections and overlaps.

August 16, 2015. Triangulation by Ear Clipping
Fixed several typographical errors and added some clarification to the constructions.

August 1, 2015. Fitting 3D Data with a Cylinder
Fixed several typographical errors and added some clarification to the constructions.

July 25, 2015. Minimum-Volume Box Containing a Set of Points
A new document describing how to compute the minimum-volume box that contains a set of 3D points.

June 23, 2015. Intersection of Ellipses
Fixed some typographical errors.

June 23, 2015. Distance from a Point to an Ellipse, an Ellipsoid, or a Hyperellipsoid
Fixed some typographical errors.

May 31, 2015. Distance to Circles in 3D
Major revision of the content of the documents for computing the distance from a 3D circle to a point, line, or circle. The documents have been combined into a single new document. The source code for the distance queries was updated to match the document, and unit tests were added internally to verify the results. The old documents have been removed, but the links to them are now redirected to the new document.

- *Distance Between Point and Circle or Disk in 3D (*DistancePoint3Circle3.pdf*)*
- *Distance Between Line and Circle or Disk in 3D (*DistanceLine3Circle3.pdf*)*
- *Distance Between Two Circles in 3D (*DistanceCircle3Circle3.pdf*)*
- *Distance Between Circle and a Disk in 3D (*DistanceCircle3Disk3.pdf*)*

May 17, 2015. Minimum-Area Rectangle Containing a Set of Points
Major revision of the document. The previous version described only the exhaustive $O(n^2)$ algorithm for computing the minimum-area rectangle bounding a convex polygon. The revised version describes the $O(n)$ rotating calipers algorithm. The GTEngine source code was modified to match the document, and it now runs nearly twice as fast as the previous code (that did not do an efficient job of maintaining the supporting vertex information).

May 13, 2015. An Approximation for the Inverse Square Root Function
Removed the document *Fast and Accurate Inverse Square Root* (FastInverseSqrt.pdf). Enough is enough for wondering where the magic number came from (see the Wikipedia page for details about fast inverse square root). Added the following document for approximating the inverse square root function using a minimax approach. The algorithm is of interest because it establishes a mathematical pattern that can be used for other function approximations. Its application to inverse square root is of interest only for hardware that does not have support for the operation (current generation floating-point units do have support).

April 25, 2015.
Removed documents that were not accessed frequently, are not particularly important, need significant rewriting, or have been rewritten and the old links redirected to the new files.

- *Bisection in 1D, 2D, and 3D* (Bisection.pdf)
- *Compressed Unit Vectors* (CompressedUnitVectors.pdf)
- *CORDIC Methods* (CordicMethods.pdf)
- *A Collinearity Test Independent of Point Order* (CollinearityTest.pdf)
- *Command Line Parsing* (CommandLineParsing.pdf)
- *Distance from Point to a General Quadratic Curve or a General Quadric Surface* (DistancePointToQuadratic.pdf)
- *Eigensystems for 3x3 Symmetric Matrices (Revisited)* (EigenSymmetric3x3.pdf)
- *Eigensystems for Symmetric Matrices* (EigenSymmetricNxN.pdf)
- *Geometric Invariance (vector fields, Lie algebra, prolongations)* (GeometricInvariance.pdf)
- *Conversion of Left-Handed Coordinates to Right-Handed Coordinates* (LeftHandedToRightHanded.pdf)
- *Numerical Methods for Ordinary Differential Equations* (OdeNumericalMethods.pdf)
- *Numerical Methods for Partial Differential Equations* (PdeNumericalMethods.pdf)
- *Reconstructing a Point from Distances* (PointFromDistances.pdf)
- *Polysolids and Boolean Operations* (Polysolids.pdf)
- *Special Functions* (SpecialFunctions.pdf)
- *Spherical Harmonics* (SphericalHarmonics.pdf)
- *Subdivision of a Parabolic Segment by Arc Length* (SubdivideParabola.pdf)

April 25, 2015. Derivative Approximations by Finite Differences
Slight clean-up of the document and a fix to several equation references.

April 24, 2015. Representing a Circle or a Sphere with NURBS
Expanded the document to include more NURBS representations. In particular, there is a degree-3 representation for a full circle that uses an internal repeated knot to splice together two half-circles. There is a

tensor product NURBS surface, degree-3 in each parameter, that uses the same idea of an internal repeated knot to splice together two hemispheres.

April 23, 2015. Converting Between Coordinate Systems
Major rewrite of the document on converting between coordinate systems. The original version of this document was entitled *Conversion of Left-Handed Coordinates to Right-Handed Coordinates* and was written to handle the conversion of LightWave coordinate systems (left-handed) to Wild Magic coordinate systems (right-handed). The process was specific to LightWave's choice of representing rotations using Euler angles, and the discussion included how to deal with cameras, lights, and transformation hierarchies. The new version is the general process of converting between any two coordinate systems. An implementation is provided that automates the process.

April 20, 2015. Intersection of Ellipses, The Area of Intersecting Ellipses
Added significant improvements to the document for the test-intersection and find-intersection queries for ellipses and to the document for computing the area of intersection of ellipses. The source code implementation for the test-intersection and find-intersection queries were updated to use the details described in the document. Implemented the query for area of intersection of ellipses.

April 4, 2015. Low-Degree Polynomial Roots
Rewrote the document for computing the roots of low-degree polynomials. The new document goes into great detail about the classification of roots (real or non-real, multiplicities) and how to use exact rational arithmetic to correctly classify the roots in a program. This leads to more robust root finding using the closed-form expressions for polynomials of degrees 2, 3, and 4. The motivation for the revisions was based on trying to compute intersections of ellipses, and the nonrobust root finder for quartic polynomials created many problems numerically.

April 2, 2015. GTEngine: Arbitrary Precision Arithmetic
Fixed the algorithm description for conversion from a BSRational object to a floating-point type. When the round-up step causes a carry-out, so to speak, from the trailing significand, a block of code was executed to set w to 1 and adjust p-q. This was incorrect and, in fact, not necessary because w is not used as the trailing significand in the conversion.

March 29, 2015. Intersection of Cylinders
Added some definitions for variables used in the equations in the section *Separation Tests Involving Other Directions*. Converted the verbatim pseudocode to use lstlisting format.

March 10, 2015. Parallel Projection of an Ellipse, Perspective Projection of an Ellipse
Fixed typographical errors and added some clarification.

January 26, 2015. Robust Computation of Distance Between Line Segments
Fixed an error in the description for region 2 and added some clarification about the sign tests for the partial derivatives of $R(s,t)$.

January 5, 2015. Least Squares Fitting of Data
Added comments about fitting lines $y = ax + b$ and planes $z = ax + by + c$, mentioning that the numerical problem for solving the linear equations in the coefficients is ill conditioned unless you compute the averages of the data points first and subtract those averages from the data points before solving the linear system.

January 4, 2015. Intersection of a Box and a Cone
A new document describing the test-intersection query for boxes and cones.

December 28, 2014. Intersection of a Line and a Cone

Modified the document on line-cone intersection to give greater detail. The source code was updated accordingly.

December 12, 2014. Distance from a Point to an Ellipse, an Ellipsoid, or a Hyperellipsoid
The document was rewritten to add more details about the algorithm. The source code was modified accordingly.

December 7, 2014. A Robust Eigensolver for 3x3 Symmetric Matrices
A new document describing a variation of an iterative eigensolver for symmetric 3x3 matrices. A source code implementation and a sample application are provided. The GTEngine code now uses only the iterative solvers for symmetric matrices; we also have an iterative implementation for the singular value decomposition (SVD).

November 27, 2014. Fitting 3D Data with a Cylinder
The pseudocode for computing the fitted cylinder subtracted the input point average for numerically stable computations. The returned center needed the average added to it.

November 25, 2014. GTEngine: Arbitrary Precision Arithmetic
Overhauled the arbitrary precision library to improve the performance and to improve readability. The unsigned integer arithmetic was factored out of BSNumber into two classes, one for arbitrary precision with storage of type std::vector and one for user-selected fixed precision with storage of type std::array. Both classes share code for the arithmetic logic unit. Many computations are now performed in-place to avoid expensive allocation, deallocation, and memory copies. A new PDF is posted that greatly expands on the library compared to the discussion in the *GPGPU Programming for Games and Science* book. The document serves as a discussion about the design of the library and a reference for how to use it. Examples are provided for using BSPrecision to determine the template parameter of UInteger<N> that represents the maximum number of bits required to compute the exact results for a sequence of expressions.

November 6, 2014. Distance Between Two Line Segments in 3D
Implemented a robust algorithm for computing the distance between line segments in any dimension. Revised the PDF for computing distance between segments in 3D to describe the new algorithm. A GPU implementation is available in the sample application.

October 20, 2014. Platonic Solids (parameters, vertices, mesh connectivity
The volume equation for a dodecahedron was in error.

August 15, 2014. The Area of Intersecting Ellipses
Fixed a typographical error in Equation (14). Replaced the LaTeX verbatim commands with lstlisting commands for more readable pseudocode.