Distance Between Two Ellipses in 3D

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1 Introduction

An ellipse in 3D is represented by a center $\mathbf{C}$, unit-length axes $\mathbf{U}$ and $\mathbf{V}$ with corresponding axis lengths a and b, and a plane containing the ellipse, $\mathbf{N}\cdot(\mathbf{X}-\mathbf{C}) = 0$ where $\mathbf{N}$ is a unit length normal to the plane. The vectors $\mathbf{U}$, $\mathbf{V}$, and $\mathbf{N}$ form a right-handed orthonormal coordinate system (the matrix with these vectors as columns is orthonormal with determinant 1). The ellipse is parameterized as

$$\mathbf{X} = \mathbf{C} + a\cos(\theta)\mathbf{U} + b\sin(\theta)\mathbf{V}$$

for angles $\theta \in [0, 2\pi)$. The ellipse is also defined by the two polynomial equations

$$\begin{aligned}&\mathbf{N}\cdot(\mathbf{X}-\mathbf{C}) = 0\\&(\mathbf{X}-\mathbf{C})^{\mathsf{T}}\left(\frac{\mathbf{U}\mathbf{U}^{\mathsf{T}}}{a^2} + \frac{\mathbf{V}\mathbf{V}^{\mathsf{T}}}{b^2}\right)(\mathbf{X}-\mathbf{C}) = 1\end{aligned}$$

where the last equation is written as a quadratic form. The first equation defines a plane and the second equation defines an ellipsoid. The intersection of plane and ellipsoid is an ellipse.

2 Solution as Polynomial System

The two ellipses are $\mathbf{N}_0\cdot(\mathbf{X}-\mathbf{C}_0) = 0$ and $(\mathbf{X}-\mathbf{C}_0)^{\mathsf{T}}A_0(\mathbf{X}-\mathbf{C}_0) = 1$ where $A_0 = \mathbf{U}_0\mathbf{U}_0^{\mathsf{T}}/a_0^2 + \mathbf{V}_0\mathbf{V}_0^{\mathsf{T}}/b_0^2$ and $\mathbf{N}_1\cdot(\mathbf{Y}-\mathbf{C}_1) = 0$ and $(\mathbf{Y}-\mathbf{C}_1)^{\mathsf{T}}A_1(\mathbf{Y}-\mathbf{C}_1) = 1$ where $A_1 = \mathbf{U}_1\mathbf{U}_1^{\mathsf{T}}/a_1^2 + \mathbf{V}_1\mathbf{V}_1^{\mathsf{T}}/b_1^2$.

The problem is to minimize the squared distance $|\mathbf{Y}-\mathbf{Y}|^2$ subject to the four constraints mentioned above. The problem can be solved with the method of Lagrange multipliers. Introduce four new parameters, α, β, γ, and δ and minimize

$$\begin{aligned}F(\mathbf{X},\mathbf{Y};\alpha,\beta,\gamma,\delta) &= |\mathbf{X}-\mathbf{Y}|^2 + \alpha((\mathbf{X}-\mathbf{C}_0)^{\mathsf{T}}A_0(\mathbf{X}-\mathbf{C}_0) - 1) + \beta(\mathbf{N}_0\cdot(\mathbf{X}-\mathbf{C}_0) - 0)\\&\quad+\gamma((\mathbf{Y}-\mathbf{C}_1)^{\mathsf{T}}A_1(\mathbf{Y}-\mathbf{C}_1) - 1) + \delta(\mathbf{N}_1\cdot(\mathbf{Y}-\mathbf{C}_1) - 0).\end{aligned}$$

Taking derivatives yields

$$\begin{aligned}F_{\mathbf{X}} &= 2(\mathbf{X}-\mathbf{Y}) + 2\alpha A_0(\mathbf{X}-\mathbf{C}_0) + \beta\mathbf{N}_0\\F_{\mathbf{Y}} &= -2(\mathbf{X}-\mathbf{Y}) + 2\gamma A_1(\mathbf{Y}-\mathbf{C}_1) + \delta\mathbf{N}_1\\F_\alpha &= (\mathbf{X}-\mathbf{C}_0)^{\mathsf{T}}A_0(\mathbf{X}-\mathbf{C}_0) - 1\\F_\beta &= \mathbf{N}_0\cdot(\mathbf{X}-\mathbf{C}_0)\\F_\gamma &= (\mathbf{Y}-\mathbf{C}_1)^{\mathsf{T}}A_1(\mathbf{Y}-\mathbf{C}_1) - 1\\F_\delta &= \mathbf{N}_1\cdot(\mathbf{Y}-\mathbf{C}_1)\end{aligned}$$

Setting the last four equations to zero yields the four original constraints. Setting the first equation to the zero vector and multiplying by $(\mathbf{X}-\mathbf{C}_0)^{\mathsf{T}}$ yields

$$\alpha = -2(\mathbf{X}-\mathbf{C}_0)^{\mathsf{T}}(\mathbf{X}-\mathbf{Y}).$$

Setting the first equation to the zero vector and multiplying by $\mathbf{N}_0^{\mathsf{T}}$ yields

$$\beta = -2\mathbf{N}_0^{\mathsf{T}}(\mathbf{X}-\mathbf{Y}).$$

Similar manipulations of the second equation yield

$$\gamma = 2(\mathbf{Y} - \mathbf{C}_1)^{\mathsf{T}}(\mathbf{X} - \mathbf{Y})$$

and

$$\delta = 2\mathbf{N}_1^{\mathsf{T}}(\mathbf{X} - \mathbf{Y}).$$

The first two derivative equations become

$$\begin{aligned}
M_0(\mathbf{X} - \mathbf{Y}) &= \left(\mathbf{N}_0\mathbf{N}_0^{\mathsf{T}} + A_0(\mathbf{X} - \mathbf{C}_0)(\mathbf{X} - \mathbf{C}_0)^{\mathsf{T}} - I\right)(\mathbf{X} - \mathbf{Y}) = \mathbf{0} \\
M_1(\mathbf{X} - \mathbf{Y}) &= \left(\mathbf{N}_1\mathbf{N}_1^{\mathsf{T}} + A_1(\mathbf{Y} - \mathbf{C}_1)(\mathbf{Y} - \mathbf{C}_1)^{\mathsf{T}} - I\right)(\mathbf{X} - \mathbf{Y}) = \mathbf{0}
\end{aligned}$$

Observe that $M_0\mathbf{N}_0 = \mathbf{0}$, $M_0A_0(\mathbf{X} - \mathbf{C}_0) = \mathbf{0}$, and $M_0(\mathbf{N}_0 \times (\mathbf{X} - \mathbf{C}_0)) = -\mathbf{N}_0 \times (\mathbf{X} - \mathbf{C}_0)$. Therefore, $M_0 = -\mathbf{W}_0\mathbf{W}_0^{\mathsf{T}}/|\mathbf{W}_0|^2$ where $\mathbf{W}_0 = \mathbf{N}_0 \times (\mathbf{X} - \mathbf{C}_0)$. Similarly, $M_1 = -\mathbf{W}_1\mathbf{W}_1^{\mathsf{T}}/|\mathbf{W}_1|^2$ where $\mathbf{W}_1 = \mathbf{N}_1 \times (\mathbf{Y} - \mathbf{C}_1)$. The previous displayed equations are equivalent to $\mathbf{W}_0^{\mathsf{T}}(\mathbf{X} - \mathbf{Y}) = 0$ and $\mathbf{W}_1^{\mathsf{T}}(\mathbf{X} - \mathbf{Y}) = 0$.

The points $\mathbf{X} = (x_0, x_1, x_2)$ and $\mathbf{Y} = (y_0, y_1, y_2)$ that attain minimum distance between the two ellipses are solutions to the six quadratic equations in six unknowns,

$$\begin{aligned}
p_0(x_0, x_1, x_2) &= \mathbf{N}_0 \cdot (\mathbf{X} - \mathbf{C}_0) = 0, \\
p_1(x_0, x_1, x_2) &= (\mathbf{X} - \mathbf{C}_0)^{\mathsf{T}}A_0(\mathbf{X} - \mathbf{C}_0) = 1, \\
p_2(x_0, x_1, x_2, y_0, y_1, y_2) &= (\mathbf{X} - \mathbf{Y}) \cdot \mathbf{N}_0 \times (\mathbf{X} - \mathbf{C}_0) = 0, \\
q_0(y_0, y_1, y_2) &= \mathbf{N}_1 \cdot (\mathbf{Y} - \mathbf{C}_1) = 0, \\
q_1(y_0, y_1, y_2) &= (\mathbf{Y} - \mathbf{C}_1)^{\mathsf{T}}A_1(\mathbf{Y} - \mathbf{C}_1) = 1, \\
q_2(x_0, x_1, x_2, y_0, y_1, y_2) &= (\mathbf{X} - \mathbf{Y}) \cdot \mathbf{N}_1 \times (\mathbf{Y} - \mathbf{C}_1) = 0.
\end{aligned}$$

On a computer algebra system that supports the resultant operation for eliminating polynomial variables, the following set of operations leads to a polynomial in one variable. Let $\mathrm{Resultant}[P, Q, z]$ denote the resultant of polynomials P and Q where the variable z is eliminated,

$$\begin{aligned}
r_0(x_0, x_1, y_0, y_1, y_2) &= \mathrm{Resultant}[p_0, p_2, x_2] \\
r_1(x_0, x_1) &= \mathrm{Resultant}[p_1, p_2, x_2] \\
r_2(x_0, x_1, y_0, y_1) &= \mathrm{Resultant}[r_0, q_2, y_2] \\
s_0(x_0, x_1, x_2, y_0, y_1) &= \mathrm{Resultant}[q_0, q_2, y_2] \\
s_1(y_0, y_1) &= \mathrm{Resultant}[q_1, q_2, y_2] \\
s_2(x_0, x_1, y_0, y_1) &= \mathrm{Resultant}[s_0, p_2, x_2] \\
r_3(x_0, y_0, x_1) &= \mathrm{Resultant}[r_2, r_1, x_1] \\
r_4(x_0, y_0) &= \mathrm{Resultant}[r_3, s_1, y_1] \\
s_3(x_0, x_1, y_0) &= \mathrm{Resultant}[s_2, s_1, y_1] \\
s_4(x_0, y_0) &= \mathrm{Resultant}[s_3, r_1, x_1] \\
\phi(x_0) &= \mathrm{Resultant}[r_4, s_4, y_0]
\end{aligned}$$

For two circles, the degree of ϕ is 8. For a circle and an ellipse, the degree of ϕ is 12. For two ellipses, the degree of ϕ is 16.

3 Solution using Trigonometric Approach

Let the two ellipses be

$$\mathbf{X} = \mathbf{C}_0 + a_0 \cos(\theta)\mathbf{U}_0 + b_0 \sin(\theta)\mathbf{V}_0$$
$$\mathbf{Y} = \mathbf{C}_1 + a_1 \cos(\phi)\mathbf{U}_1 + b_1 \sin(\phi)\mathbf{V}_1$$

for $\theta \in [0, 2\pi)$ and $\phi \in [0, 2\pi)$. The squared distance between any two points on the ellipses is $F(\theta, \phi) = |\mathbf{X}(\theta) - \mathbf{Y}(\phi)|^2$. The problem is to minimize $F(\theta, \phi)$.

Define $c_0 = \cos(\theta)$, $s_0 = \sin(\theta)$, $c_1 = \cos(\phi)$, and $s_1 = \sin(\phi)$. Compute derivatives, $F_\theta = (\mathbf{X}(\theta) - \mathbf{Y}(\phi)) \cdot \mathbf{X}'(\theta)$ and $F_\phi = -(\mathbf{X}(\theta) - \mathbf{Y}(\phi)) \cdot \mathbf{Y}'(\phi)$. Setting these equal to zero leads to the two polynomial equations in c_0, s_0, c_1, and s_1. The two polynomial constraints for the sines and cosines are also listed.

$$p_0 = (a_0^2 - b_0^2)s_0c_0 + a_0(\alpha_{00} + \alpha_{01}s_1 + \alpha_{02}c_1)s_0 + b_0(\beta_{00} + \beta_{01}s_1 + \beta_{02}c_1)c_0 = 0$$
$$p_1 = (a_1^2 - b_1^2)s_1c_1 + a_1(\alpha_{10} + \alpha_{11}s_0 + \alpha_{12}c_0)s_1 + b_1(\beta_{10} + \beta_{11}s_0 + \beta_{12}c_0)c_1 = 0$$
$$q_0 = s_0^2 + c_0^2 - 1 = 0$$
$$q_1 = s_1^2 + c_1^2 - 1 = 0$$

This is a system of four quadratic polynomial equations in four unknowns and can be solved with resultants:

$$\begin{aligned}
r_0(s_0, s_1, c_1) &= \text{Resultant}[p_0, q_0, c_0] \\
r_1(s_0, s_1, c_0) &= \text{Resultant}[p_1, q_1, c_1] \\
r_2(s_0, s_1) &= \text{Resultant}[r_0, q_1, c_1] \\
r_3(s_0, s_1) &= \text{Resultant}[r_1, q_0, c_0] \\
\phi(s_0) &= \text{Resultant}[r_2, r_3, s_1]
\end{aligned}$$

Alternatively, we can use the simple nature of q_0 and q_1 to do some of the elimination. Let $p_0 = \alpha_0 s_0 + \beta_0 c_0 + \gamma_0 s_0 c_0$ where α_0 and β_0 are linear polynomials in s_1 and c_1. Similarly, $p_1 = \alpha_1 s_1 + \beta_1 c_1 + \gamma_1 s_1 c_1$ where α_1 and β_1 are linear polynomials in s_0 and c_0. Solving for c_0 in $p_0 = 0$ and c_1 in $p_1 = 0$, squaring, and using the q_i constraints leads to

$$r_0 = (1 - s_0^2)(\gamma_0 s_0 + \beta_0)^2 - \alpha_0^2 s_0^2 = 0$$
$$r_1 = (1 - s_1^2)(\gamma_1 s_1 + \beta_1)^2 - \alpha_1^2 s_1^2 = 0$$

Using the q_i constraints, write $r_i = r_i0 + r_i1c_{1-i}$, $i = 0, 1$, where the r_{ij} are polynomials in s_0 and s_1. The terms r_i0 are degree 4 and the terms r_{i1} is degree 3. Solving for c_0 in $r_1 = 0$ and c_1 in $r_0 = 0$, squaring, and using the q_i constraints leads to

$$w_0 = (1 - s_1^2)r_{01}^2 - r_{00}^2 = \textstyle\sum_{j=0}^{8} w_{0j} s_0^j = 0$$
$$w_1 = (1 - s_0^2)r_{11}^2 - r_{10}^2 = \textstyle\sum_{j=0}^{4} w_{1j} s_0^j = 0$$

The coefficients w_{ij} are polynomials in s_1. The degrees of w_{00} through w_{08} respectively are 4, 3, 4, 3, 4, 3, 2, 1, and 0. The degree of w_{1j} is $8 - j$. Total degree for each of w_i is 8.

The final elimination can be computed using a Bezout determinant, $\phi(s_1) = \det[e_{ij}]$, where the underlying matrix is $8x8$ and the entry is

$$e_{ij} = \sum_{k=\max(9-i,9-j)}^{\min(8,17-i-j)} v_{k,17-i-j-k}$$

where $v_{i,j} = w_{0i}w_{1j} - w_{0j}w_{1i}$. If the i or j index is out of range in the w terms, then the term is assumed to be zero. The solutions to $\phi = 0$ are the candidate points for s_1. For each s_1, two c_1 values are computed using $s_1^2 + c_1^2 = 1$. For each s_1, the roots of the polynomial $w_1(s_0)$ are computed. For each s_0, two c_0 values are computed using $s_0^2 + c_0^2 = 1$. Out of all such candidates, $|\mathbf{X} - \mathbf{Y}|^2$ can be computed and the minimum value is selected.

4 Numerical Solution

Neither algebraic method above seems reasonable. Each looks very slow to compute and you have the usual numerical problems with polynomials of large degree. I have not implemented the following, but my guess is it is an alternative to consider. Implement a distance calculator for point to ellipse (in 3D). This is a function of a single parameter, say $F(\theta)$ for $\theta \in [0, 2\pi]$. Use a numerical minimizer that does not require derivative calculation (Powell's method for example) and minimize F on the interval $[0, 2\pi]$. The scheme is iterative and hopefully converges rapidly to the solution.