Classifying Quadrics using Exact Arithmetic

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Contents

1 Introduction

A quadratic equation in three variables is

$$\boldsymbol{x}^{\mathsf{T}} A \boldsymbol{x} + \boldsymbol{b}^{\mathsf{T}} \boldsymbol{x} + c = 0 \tag{1}$$

where

$$\boldsymbol{x} = \begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix}, \quad A = \begin{bmatrix} a_{00} & a_{01} & a_{02} \\ a_{01} & a_{11} & a_{12} \\ a_{02} & a_{12} & a_{22} \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} b_0 \\ b_1 \\ b_2 \end{bmatrix} \tag{2}$$

The problem is to classify the solution set to the quadratic equation. In many applications, we are interested only in ellipsoids, hyperboloids or paraboloids. However, a quadratic equation can define other surfaces including cylinder surfaces and planes. It is even possible to generate lines and a point via a quadratic. And it is possible that there is no solution to the equation.

A methodical approach is presented here for the classification. This is a mathematical construction assuming real numbers. An implementation that uses floating-point arithmetic can suffer from rounding errors, causing misclassifications. If the coefficients of the quadratic equation are converted to their rational number equivalents, exact arithmetic can be used to classify the solution set without errors. Be aware that if the coefficients are outputs from a computational process that uses floating-point arithmetic, the coefficients might already have rounding errors that can change the classifications. The rational-based classification is correct only for the input floating-point coefficients.

2 Signs of the Eigenvalues of A

The matrix A is symmetric, so it has an eigendecomposition $A = R \Lambda R^{\mathsf{T}}$ where $R = [\boldsymbol{v}_0 \mid \boldsymbol{v}_1 \mid \boldsymbol{v}_2]$ is a rotation matrix whose real-valued columns $\boldsymbol{v}_i$ are linearly independent eigenvectors of A and where $\Lambda = \mathrm{Diag}(\lambda_0, \lambda_1, \lambda_2)$ is a real-valued diagonal matrix of eigenvalues of A. The eigenvector $\boldsymbol{v}_i$ corresponds to the eigenvalue λ_i.

2.1 Characteristic Polynomial of A

The eigenvalues are roots of the characteristic polynomial of A, which is the cubic polynomial

$$p(\lambda) = \det(\lambda I - A) = \lambda^3 + p_2 \lambda^2 + p_1 \lambda + p_0 \tag{3}$$

where I is the 3×3 identity matrix. The real-valued coefficients are

$$\begin{aligned} p_0 &= -\left(a_{00}(a_{11}a_{22} - a_{12}^2) - a_{01}(a_{01}a_{22} - a_{12}a_{02}) + a_{02}(a_{01}a_{12} - a_{02}a_{11})\right) = -\,\mathrm{Determinant}(A) \\ p_1 &= (a_{00}a_{11} - a_{01}^2) + (a_{00}a_{22} - a_{02}^2) + (a_{11}a_{22} - a_{12}^2) \\ p_2 &= -\,(a_{00} + a_{11} + a_{22}) = -\,\mathrm{Trace}(A) \end{aligned} \tag{4}$$

The polynomial has only real-valued roots. Closed-form equations exist for the roots. When the roots are distinct, the roots are generally not rational numbers. If A has rational components, the roots are generally not rational numbers. However, when at least one root is repeated, the roots are rational.

If A and $\boldsymbol{b}$ have rational components and c is rational, obtaining an exact classification of the solution set for the quadratic equation is possible but requires formulations that ensure various expressions are rational valued. In the discussion, assume the eigenvalues are ordered as $\lambda_0 \leq \lambda_1 \leq \lambda_2$.

2.2 Descartes' Rule of Signs

Descartes' rule of signs [1] applies to polynomials $p(\lambda)$ with real-valued coefficients. If the nonzero real-valued coefficients of $p(\lambda)$ are ordered by descending exponent, then the number of positive roots is either the number of sign changes of the ordered coefficients or the number of sign changes minus an even number. A root of multiplicity m is counted as m roots. When the number of positive roots is less than the number of sign changes, the polynomial has pairs of non-real roots that are complex conjugates. The number of negative roots of $p(\lambda)$ is the number of positive roots of $p(-\lambda)$.

The characteristic polynomial $p(\lambda)$ for a real-valued symmetric matrix A has only real-valued roots. If A is $n \times n$, the k distinct roots are r_i for $0 \leq i < k$, each having multiplicity m_i. The sum of the multiplicities is n. Descartes' rule of signs produces the exact number of positive roots, say, n_+, and the exact number of negative roots, say, n_-. Let n_0 be the number of zero roots; then $n_+ + n_i + n_0 = n$. Listing 1 contains pseudocode for computing the numbers of roots of $p(\lambda)$.

Listing 1. Pseudocode for Descartes' rule of signs to count the number of positive roots, the number of negative roots and the number of zero roots for the characteristic polynomial $p(\lambda)$. The pseudocode is valid regardless whether the characteristic polynomial is defined by $p(\lambda) = \det(\lambda I - A)$ or by $p(\lambda) = \det(A - \lambda_i)$.

```
void ComputeNumberRoots(Polynomial p, int& numPositiveRoots, int& numNegativeRoots, int& numZeroRoots)
{
    // Assert p(lambda) = det(lambda * I - A).
    // Assert n >= 1 and p[n] is not zero.
    int n = Degree(p);

    // Compute the signs of the coefficients of p(lambda). The values are in {-1,0,1}.
    int sign[n + 1];
    for (int i = 0; i <= n; ++i)
    {
        sign[i] = Sign(p[i]);
    }

    // Compute the number of positive roots of p(lambda).
    int currentSign = sign[n];
    numPositiveRoots = 0;
    for (int i = n-1; i >= 0; --i)
    {
        int s = sign[i];
        if (sign == -currentSign)
        {
            currentSign = -1;
            ++numPositiveRoots;
        }
    }

    // Compute the signs of the coefficients of p(-lambda).
    currentSign = 1;
    for (int i = 0; i <= n; ++i)
    {
        sign[i] *= currentSign;
        currentSign = -currentSign;
    }

    // Compute the number of negative roots of p(lambda).
    currentSign = sign[n];
```

```
    numNegativeRoots = 0;
    for (int i = n-1; i >= 0; --i)
    {
        int s = sign[i];
        if (sign == -currentSign)
        {
            currentSign = -1;
            ++numNegativeRoots;
        }
    }

    // Compute the number of zero roots of p(lambda).
    numZeroRoots = n - numPositiveRoots - numNegativeRoots;
}
```

3 Classification of the Solution Set of the Quadratic Equation

3.1 All Nonzero Eigenvalues

The matrix A is invertible because all eigenvalues are not zero. Define $\boldsymbol{u} = -A^{-1}\boldsymbol{b}/2$. Equation (1) factors into $(\boldsymbol{x}-\boldsymbol{u})^{\mathsf{T}}A(\boldsymbol{x}-\boldsymbol{u}) = \boldsymbol{u}^{\mathsf{T}}A\boldsymbol{u} - c$. Using the eigendecomposition $A = R\Lambda R^{\mathsf{T}}$ and defining $\boldsymbol{y} = R^{\mathsf{T}}(\boldsymbol{x}-\boldsymbol{u})$, the equation is further modified to

$$\lambda_0 y_0^2 + \lambda_1 y_1^2 + \lambda_2 y_2^2 = \boldsymbol{y}^{\mathsf{T}}\Lambda\boldsymbol{y} = \frac{1}{4}\boldsymbol{b}^{\mathsf{T}}A^{-1}\boldsymbol{b} - c = r \tag{5}$$

where the last equality defines r. Observe that r is rational whenever A and $\boldsymbol{b}$ have rational components and c is rational. Table 1 shows the signs of the relevant values and the classifications determined by them.

Table 1. Classifications when all eigenvalues are not zero.

λ_0	λ_1	λ_2	r	classification
+	+	+	+	ellipsoid
−	+	+	+	hyperboloid of one sheet
−	−	+	+	hyperboloid of two sheets
−	−	−	+	no solution
+	+	+	−	no solution
−	+	+	−	hyperboloid of two sheets
−	−	+	−	hyperboloid of one sheet
−	−	−	−	ellipsoid
+	+	+	0	point
−	+	+	0	elliptic cone
−	−	+	0	elliptic cone
−	−	−	0	point

Listing 2 contains pseudocode for the classification when all eigenvalues are not zero.

Listing 2. Pseudocode for the classification of the solution set to Equation (1) when all eigenvalues are not zero.

```
Classification ClassifyAllNonzeroEigenvalues(
    Matrix3x3<Rational> A, Vector3<Rational> b, Rational c,
    int numPositiveLambda, int numNegativeLambda)
{
    Rational r = Dot(b, Inverse(A) * b) / 4 − c;
    if (numPositiveLambda == 3)
    {
        return (r > 0 ? ELLIPSOID : (r < 0 ? NO_SOLUTION : POINT));
    }
    else if (numPositiveLambda == 2)
    {
        return (r > 0 ? HYPERBOLOID_ONE_SHEET : (r < 0 ? HYPERBOLOID_TWO_SHEETS : ELLIPTIC_CONE));
    }
    else if (numPositiveLambda == 1)
    {
        return (r > 0 ? HYPERBOLOID_TWO_SHEETS : (r < 0 ? HYPERBOLOID_ONE_SHEET : ELLIPTIC_CONE));
    }
    else
    {
        return (r != 0 ? ELLIPTIC_CONE : POINT);
    }
}
```

3.2 Two Nonzero Eigenvalues

3.2.1 Two Positive Eigenvalues

Consider the case when $\lambda_0 = 0 < \lambda_1 \le \lambda_2$. The rank of A is 2 because it has an eigenvalue 0 of multiplicity 1. Let the 2 linearly independent rows be $\boldsymbol{\rho}_1$ and $\boldsymbol{\rho}_2$. An eigenvector for eigenvalue 0 is $\boldsymbol{w}_0 = \boldsymbol{\rho}_1 \times \boldsymbol{\rho}_2$ and is not necessarily unit length. If the A has rational components, then $\boldsymbol{w}_0$ has rational components. Choose a rational-valued vector $\boldsymbol{w}_1$ that is perpendicular to $\boldsymbol{w}_0$. Define $\boldsymbol{w}_2 = \boldsymbol{w}_0 \times \boldsymbol{w}_1$. The set $\{\boldsymbol{w}_0, \boldsymbol{w}_1, \boldsymbol{w}_2\}$ is an orthogonal set but not an orthonormal set; that is, the vectors are mutually perpendicular but they are not necessarily unit length.

Define the matrix $W = [\boldsymbol{w}_0\ \boldsymbol{w}_1\ \boldsymbol{w}_2]$ whose columns are the specified vector. Define $\boldsymbol{x} = W\boldsymbol{y}$ and $\boldsymbol{d} = W^{\mathsf{T}}\boldsymbol{b}$. Equation (1) becomes

$$\begin{aligned} 0 &= \boldsymbol{y}^{\mathsf{T}} W^{\mathsf{T}} A W \boldsymbol{y} + \boldsymbol{d}^{\mathsf{T}}\boldsymbol{y} + c \\ &= \left[\begin{array}{c|cc} y_0 & y_1 & y_2 \end{array}\right] \left[\begin{array}{c|cc} 0 & 0 & 0 \\ \hline 0 & \boldsymbol{w}_1^{\mathsf{T}} A \boldsymbol{w}_1 & \boldsymbol{w}_1^{\mathsf{T}} A \boldsymbol{w}_2 \\ 0 & \boldsymbol{w}_2^{\mathsf{T}} A \boldsymbol{w}_1 & \boldsymbol{w}_2^{\mathsf{T}} A \boldsymbol{w}_2 \end{array}\right] \left[\begin{array}{c} y_0 \\ \hline y_1 \\ y_2 \end{array}\right] + \left[\begin{array}{c|cc} d_0 & d_1 & d_2 \end{array}\right] \left[\begin{array}{c} y_0 \\ \hline y_1 \\ y_2 \end{array}\right] + c \end{aligned} \tag{6}$$

Observe that d_0 is rational whenever the components of $\boldsymbol{w}_0$ and $\boldsymbol{b}$ are rational.

If $d_0 \neq 0$, Equation (6) represents an elliptic paraboloid even if d_1 or d_2 are not rational numbers.

If $d_0 = 0$, Equation (6) has no dependence on y_0, so it is a quadratic equation in the 2 variables y_1 and y_2,

$$\begin{bmatrix} y_1 & y_2 \end{bmatrix} E \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} d_1 & d_2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + c = 0 \tag{7}$$

where E is the lower-right 2×2 block of $W^\mathsf{T} A W$. The matrices A and $W^\mathsf{T} A W$ are congruent matrices [2], so by Sylvester's law of inertia they have the same numbers of positive, negative and zero eigenvalues. Therefore, the matrix E has 2 positive eigenvalues and its eigendecomposition is $E = Q\,\mathrm{Diagonal}(\mu_1, \mu_2) Q^\mathsf{T}$ where Q is a rotation matrix and $0 < \mu_1 \le \mu_2$.

Define $\boldsymbol{\eta}^\mathsf{T} = [y_1\ y_2]$, $\boldsymbol{f}^\mathsf{T} = [d_1\ d_2]$, $\boldsymbol{k} = -E^{-1}\boldsymbol{f}/2$ and $\boldsymbol{z} = Q^\mathsf{T}(\boldsymbol{\eta} - \boldsymbol{k})$ with $\boldsymbol{z}^\mathsf{T} = [z_1\ z_2]$. Define $z_0 = y_0$. Equation (6) becomes

$$\mu_1 z_1^2 + \mu_2 z_2^2 + d_0 z_0 = \frac{1}{4}\boldsymbol{f}^\mathsf{T} E^{-1} \boldsymbol{f}/4 - c = r \tag{8}$$

where the last equality defines r, a rational number because E and $\boldsymbol{f}$ have rational components and c is a rational number. Table 2 shows the signs of the relevant values and the classifications determined by them.

Table 2. Classifications for $\lambda_0 = 0 < \lambda_1 \le \lambda_2$. Cells without signs indicate the value is irrelevant. Cells with $\pm$ indicate the value is not zero.

d_0	r	classification
$\pm$		elliptic paraboloid
0	+	elliptic cylinder
0	0	line
0	−	no solution

Listing 3 contains pseudocode for the classification when $\lambda_0 = 0 < \lambda_1 \le \lambda_2$.

Listing 3. Pseudocode for the classification of the solution set to Equation (1) when $\lambda_0 = 0 < \lambda_1 \le \lambda_2$.

```
// Assert: Matrix A has rank 2.
void GetLinearlyIndependentRows(
    Matrix3x3<Rational> A, Vector3<Rational>& row0, Vector3<Rational>& row1)
{
    Vector3<Rational> vzero = { 0, 0, 0 };
    row0 = { A(0, 0), A(0, 1), A(0, 2) };
    if (row0 != vzero)
    {
        row1 = { A(1, 0), A(1, 1), A(1, 2) };
        if (Cross(row0, row1) == vzero)
        {
            row1 = { A(2, 0), A(2, 1), A(2, 2) };
        }
    }
    else
    {
        row0 = { A(1, 0), A(1, 1), A(1, 2) };
        row1 = { A(2, 0), A(2, 1), A(2, 2) };
    }
}
```

```
void GetOrthogonalComplement(
    Vector3<Rational> w0, Vector3<Rational>& w1, Vector3<Rational>& w2)
{
    if (abs(w0[0]) > abs(w0[1]))
    {
        w1 = { -w0[2], 0, +w0[0] };
    }
    else
    {
        w1 = { 0, +w0[2], -w0[1] };
    }
    w2 = Cross(w0, w1);
}

Classification ClassifyTwoNonzeroEigenvaluesLambda0IsZero(
    Matrix3x3<Rational> A, Vector3<Rational> b, Rational c)
{
    Vector3<Rational> row0, row1;
    GetLinearlyIndependentRows(A, row0, row1);
    Vector3<Rational> w0 = Cross(row0, row1);
    Rational d0 = Dot(w0, b);
    if (d0 != 0)
    {
        return ELLIPTIC_PARABOLOID;
    }

    Vector3<Rational> w1, w2;
    GetOrthogonalComplement(w0, w1, w2);
    Vector3<Rational> Aw1 = A * w1, Aw2 = A * w2;
    Matrix2x2<Rational> E;
    E(0, 0) = Dot(w1, Aw1);
    E(0, 1) = Dot(w1, Aw2);
    E(1, 0) = E(0, 1);
    E(1, 1) = Dot(w2, Aw2);
    Vector2<Rational> f = { Dot(w1, b), Dot(w2, b) };
    Rational r = Dot(f, Inverse(E) * f) / 4 - c;
    return (r > 0 ? ELLIPTIC_CYLINDER : (r < 0 ? NO_SOLUTION : LINE));
}
```

3.2.2 One Positive Eigenvalue and One Negative Eigenvalue

A similar formulation applies when $\lambda_0 < \lambda_1 = 0 < \lambda_2$ but leads to $\mu_1 < 0 < \mu_2$. The final quadratic equation is

$$\mu_0 z_0^2 + \mu_2 z_2^2 + d_1 z_1 = \frac{1}{4}\boldsymbol{f}^{\mathsf{T}} E^{-1} \boldsymbol{f}/4 - c = r \tag{9}$$

where E, $\boldsymbol{f}$ and r are computed for this specific case; that is, they are not the same quantities for the case $\lambda_0 = 0$. Table 3 shows the reduced equation and the signs of the relevant values and the classifications determined by them.

Table 3. Classifications for $\lambda_0 < \lambda_1 = 0 < \lambda_2$. Cells without signs indicate the value is irrelevant. Cells with $\pm$ indicate the value is not zero.

d_1	r	classification
$\pm$		hyperbolic paraboloid
0	$\pm$	hyperbolic cylinder
0		two planes

Listing 4 contains pseudocode for the classification when $\lambda_0 < \lambda_1 = 0 < \lambda_2$.

Listing 4. Pseudocode for the classification of the solution set to Equation (1) when $\lambda_0 < \lambda_1 = 0 < \lambda_2$.

```
Classification ClassifyTwoNonzeroEigenvaluesLambda1IsZero(
    Matrix3x3<Rational> A, Vector3<Rational> b, Rational c)
{
    Vector3<Rational> row0, row1;
    GetLinearlyIndependentRows(A, row0, row1);
    Vector3<Rational> w1 = Cross(row0, row1);
    Rational d1 = Dot(w1, b);
    if (d1 != 0)
    {
        return HYPERBOLIC_PARABOLOID;
    }

    Vector3<Rational> w0, w2;
    GetOrthogonalComplement(w1, w2, w0);
    Vector3<Rational> Aw0 = A * w0, Aw2 = A * w2;
    Matrix2x2<Rational> E;
    E(0, 0) = Dot(w0, Aw0);
    E(0, 1) = Dot(w0, Aw2);
    E(1, 0) = E(0, 1);
    E(1, 1) = Dot(w2, Aw2);
    Vector2<Rational> f = { Dot(w0, b), Dot(w2, b) };
    Rational r = Dot(f, Inverse(E) * f) / 4 - c;
    return (r != 0 ? HYPERBOLIC_CYLINDER : TWO_PLANES);
}
```

3.2.3 Two Negative Eigenvalues

A similar formulation also applies when $\lambda_0 \le \lambda_1 < \lambda_2 = 0$ but leads to $\mu_1 \le \mu_2 < 0$. The final quadratic equation is

$$\mu_0 z_0^2 + \mu_1 z_1^2 + d_2 z_2 = \frac{1}{4} \boldsymbol{f}^\mathsf{T} E^{-1} \boldsymbol{f}/4 - c = r \tag{10}$$

where E, $\boldsymbol{f}$ and r are computed for this specific case; that is, they are not the same quantities for the case $\lambda_0 = 0$ or for the case $\lambda_1 = 0$. Table 4 shows the signs of the relevant values and the classifications determined by them.

Table 4. Classifications when $\lambda_0 \le \lambda_1 < \lambda_2 = 0$. Cells without signs indicate the value is irrelevant. Cells with $\pm$ indicate the value is not zero.

d_2	r	classification
$\pm$		elliptic paraboloid
0	$-$	elliptic cylinder
0	0	line
0	$+$	no solution

Listing 5 contains pseudocode for the classification when $\lambda_0 \le \lambda_1 < \lambda_2 = 0$.

Listing 5. Pseudocode for the classification of the solution set to Equation (1) when $\lambda_0 \le \lambda_1 < \lambda_2 = 0$.

```
// Assert: Matrix A has rank 2.
Classification ClassifyTwoNonzeroEigenvaluesLambda2IsZero(
    Matrix3x3<Rational> A, Vector3<Rational> b, Rational c)
{
    Vector3<Rational> row0, row1;
    GetLinearlyIndependentRows(A, row0, row1);
    Vector3<Rational> w2 = Cross(row0, row1);
    Rational d2 = Dot(w2, b);
    if (d2 != 0)
    {
        return ELLIPTIC_PARABOLOID;
    }

    Vector3<Rational> w0, w1;
    GetOrthogonalComplement(w2, w0, w1);
    Vector3<Rational> Aw0 = A * w0, Aw1 = A * w1;
    Matrix2x2<Rational> E;
    E(0, 0) = Dot(w0, Aw0);
    E(0, 1) = Dot(w0, Aw1);
    E(1, 0) = E(0, 1);
    E(1, 1) = Dot(w1, Aw1);
    Vector2<Rational> f = { Dot(w0, b), Dot(w1, b) };
    Rational r = Dot(f, Inverse(E) * f) / 4 - c;
    return (r < 0 ? ELLIPTIC_CYLINDER : (r > 0 ? NO_SOLUTION : LINE));
}
```

3.3 One Nonzero Eigenvalue

3.3.1 Positive Eigenvalue

Consider the case when $\lambda_0 = \lambda_1 = 0 < \lambda_2$. The rank of A is 1 because it has an eigenvalue 0 of multiplicity 2. Let the nonzero row be $\boldsymbol{w}_2$. It is an eigenvector for the nonzero eigenvalue; it has rational components but is not necessarily unit length. Choose a rational-valued vector $\boldsymbol{w}_0$ that is perpendicular to $\boldsymbol{w}_2$ and define $\boldsymbol{w}_1 = \boldsymbol{w}_2 \times \boldsymbol{w}_0$. The vectors $\boldsymbol{w}_0$ and $\boldsymbol{w}_1$ are linearly independent eigenvectors for eigenvalue 0.

Define the matrix $W = [\boldsymbol{w}_0\, \boldsymbol{w}_1\, \boldsymbol{w}_2]$ whose columns are the specified vectors. Define $\boldsymbol{x} = W\boldsymbol{y}$ and $\boldsymbol{d} = W^{\mathsf{T}}\boldsymbol{b}$. Equation (1) becomes

$$\begin{aligned} 0 &= \boldsymbol{y}^{\mathsf{T}} W^{\mathsf{T}} A W \boldsymbol{y} + \boldsymbol{d}^{\mathsf{T}}\boldsymbol{y} + c \\ &= \left[\begin{array}{cc|c} y_0 & y_1 & y_2 \end{array}\right] \left[\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & \boldsymbol{w}_2^{\mathsf{T}} A \boldsymbol{w}_2 \end{array}\right] \left[\begin{array}{c} y_0 \\ y_1 \\ \hline y_2 \end{array}\right] + \left[\begin{array}{cc|c} d_0 & d_1 & d_2 \end{array}\right] \left[\begin{array}{c} y_0 \\ y_1 \\ \hline y_2 \end{array}\right] + c \end{aligned} \tag{11}$$

Observe that d_0 and d_1 are rational numbers.

If $d_0 \ne 0$ or $d_1 \ne 0$, Equation (11) represents a parabolic cylinder.

If $d_0 = d_1 = 0$, Equation (11) has no dependence on y_0 or y_1, so it is a quadratic equation in the single variable y_2 which can be factored to

$$e_2 \left(y_2 + d_2/(2e_2)\right)^2 = d_2^2/(4e_2) - c = r \tag{12}$$

where $e_2 = \boldsymbol{w}_2^\mathsf{T} A \boldsymbol{w}_2 = \lambda_2 |\boldsymbol{w}_2|^2$. Define $z_0 = y_0$, $z_1 = y_1$ and $z_2 = y_2 + d_2/(2e_2)$ to obtain the general equation handling all cases of d_0 and d_1,

$$e_2 z_2^2 + d_0 z_0 + d_1 z_1 = r \tag{13}$$

Table 5 shows the signs of the relevant values and the classifications determined by them.

Table 5. Classifications for $\lambda_0 = \lambda_1 = 0 < \lambda_2$. Cells without signs indicate the value is irrelevant. Cells with $\pm$ indicate the value is not zero.

d_0	d_1	r	classification
$\pm$			parabolic cylinder
	$\pm$		parabolic cylinder
0	0	$+$	two planes
0	0	0	one plane
0	0	$-$	no solution

Listing 6 contains pseudocode for the classification when $\lambda_0 = \lambda_1 = 0 < \lambda_2$.

Listing 6. Pseudocode for the classification of the solution set to Equation (1) when $\lambda_0 = \lambda_1 = 0 < \lambda_2$.

```
void GetNonzeroRow(Matrix3x3<Rational> A, Vector3<Rational>& row2)
{
    Vector3<Rational> vzero = { 0, 0, 0 };
    row2 = { A(0, 0), A(0, 1), A(0, 2) }
    if (row2 == vzero)
    {
        row2 = { A(1, 0), A(1, 1), A(1, 2) };
        if (row2 == vzero)
        {
            row2 = { A(2, 0), A(2, 1), A(2, 2) };
        }
    }
}

Classification ClassifyOneNonzeroEigenvaluesLambda2IsPositive(
    Matrix3x3<Rational> A, Vector3<Rational> b, Rational c)
{
    Vector3<Rational> w2;
    GetNonzeroRow(A, w2);
    Vector3<Rational> w0, w1;
    GetOrthogonalComplement(w2, w0, w1);
    Rational d0 = Dot(w0, b);
    if (d0 != 0)
    {
        return ELLIPTIC_PARABOLOID;
    }
    Rational d1 = Dot(w1, b);
    if (d1 != 0)
    {
        return ELLIPTIC_PARABOLOID;
    }

    Rational e2 = Dot(w2, A * w2);
    Rational d2 = Dot(w2, b);
    Rational r = d2 * d2 / (4 * e2) - c;
    return (r > 0 ? TWO_PLANES: (r < 0 ? NO_SOLUTION : ONE_PLANE));
}
```

3.3.2 Negative Eigenvalue

A similar formulation also applies when $\lambda_0 < 0 = \lambda_1 = \lambda_2$ but leads to $e_0 = \lambda_0 |\boldsymbol{w}_0|^2 < 0$. The final quadratic equation is

$$e_0 z_0^2 + d_1 z_1 + d_2 z_2 = d_0^2/(4e_0) - c = r \tag{14}$$

Table 6 shows the signs of the relevant values and the classifications determined by them.

Table 6. Classifications for $\lambda_0 <= \lambda_1 = \lambda_2$. Cells without signs indicate the value is irrelevant. Cells with $\pm$ indicate the value is not zero.

d_1	d_2	r	classification
$\pm$			parabolic cylinder
	$\pm$		parabolic cylinder
0	0	$-$	two planes
0	0	0	one plane
0	0	$+$	no solution

Listing 7 contains pseudocode for the classification when $\lambda_0 < \lambda_1 = \lambda_2 = 0$.

Listing 7. Pseudocode for the classification of the solution set to Equation (1) when $\lambda_0 < \lambda_1 = \lambda_2 = 0$.

```
Classification ClassifyOneNonzeroEigenvaluesLambda0IsNegative(
    Matrix3x3<Rational> A, Vector3<Rational> b, Rational c)
{
    Vector3<Rational> w0;
    GetNonzeroRow(A, w0);
    Vector3<Rational> w1, w2;
    GetOrthogonalComplement(w0, w1, w2);
    Rational d1 = Dot(w1, b);
    if (d1 != 0)
    {
        return ELLIPTIC_PARABOLOID;
    }
    Rational d2 = Dot(w2, b);
    if (d2 != 0)
    {
        return ELLIPTIC_PARABOLOID;
    }

    Rational e0 = Dot(w0, A * w0);
    Rational d0 = Dot(w0, b);
    Rational r = d0 * d0 / (4 * e0) - c;
    return (r < 0 ? TWO_PLANES: (r > 0 ? NO_SOLUTION : ONE_PLANE));
}
```

3.4 All Zero Eigenvalues

The quadratic equation degenerates to the linear equation $b_0 x_0 + b_1 x_1 + b_2 x_2 + c = 0$. Table 7 shows the signs of the relevant values and the classifications determined by them.

Table 7. Classifications all eigenvalues are zero. Cells without signs indicate the value is irrelevant. Cells with ± indicate the value is not zero.

b_0	b_1	b_2	c	classification
±				plane
	±			plane
		±		plane
0	0	0	±	no solution
0	0	0	0	all points are solutions

References

[1] Wikipedia. Descartes' rule of signs.
https://en.wikipedia.org/wiki/Descartes'_rule_of_signs.
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