Classifying Quadrics using Exact Arithmetic

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Contents

1 Introduction

A quadratic equation in three variables is

$$\boldsymbol{x}^{\mathsf{T}} A \boldsymbol{x} + \boldsymbol{b}^{\mathsf{T}} \boldsymbol{x} + c = 0 \tag{1}$$

where

$$\boldsymbol{x} = \begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix}, \quad A = \begin{bmatrix} a_{00} & a_{01} & a_{02} \\ a_{01} & a_{11} & a_{12} \\ a_{02} & a_{12} & a_{22} \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} b_0 \\ b_1 \\ b_2 \end{bmatrix} \tag{2}$$

The problem is to classify the solution set to the quadratic equation. In many applications, we are interested only in ellipsoids, hyperboloids or paraboloids. However, a quadratic equation can define other surfaces including cylinder surfaces and planes. It is possible to generate lines and a point via a quadratic. And it is possible that there is no solution to the equation.

A methodical approach is presented here for the classification. This is a mathematical construction assuming real numbers. An implementation that uses floating-point arithmetic can suffer from rounding errors, causing misclassifications. If the coefficients of the quadratic equation are converted to their rational number equivalents, exact arithmetic can be used to classify the solution set without errors. Be aware that if the coefficients are outputs from a computational process that uses floating-point arithmetic, the coefficients might already have rounding errors that can change the classifications. The rational-based classification is correct only for the input floating-point coefficients.

2 Signs of the Eigenvalues of A

The matrix A is symmetric, so it has an eigendecomposition $A = R \Lambda R^{\mathsf{T}}$ where $R = [\boldsymbol{v}_0 \mid \boldsymbol{v}_1 \mid \boldsymbol{v}_2]$ is a rotation matrix whose real-valued columns $\boldsymbol{v}_i$ are linearly independent eigenvectors of A and where $\Lambda = \mathrm{Diag}(\lambda_0, \lambda_1, \lambda_2)$ is a real-valued diagonal matrix of eigenvalues of A. The eigenvector $\boldsymbol{v}_i$ corresponds to the eigenvalue λ_i.

2.1 Characteristic Polynomial of A

The eigenvalues are roots of the characteristic polynomial of A, which is the cubic polynomial

$$p(\lambda) = \det(\lambda I - A) = \lambda^3 + p_2 \lambda^2 + p_1 \lambda + p_0 \tag{3}$$

where I is the 3×3 identity matrix. The real-valued coefficients are

$$
\begin{aligned}
p_0 &= -\left(a_{00}(a_{11}a_{22} - a_{12}^2) - a_{01}(a_{01}a_{22} - a_{12}a_{02}) + a_{02}(a_{01}a_{12} - a_{02}a_{11}) \right) = -\,\mathrm{Determinant}(A) \\
p_1 &= (a_{00}a_{11} - a_{01}^2) + (a_{00}a_{22} - a_{02}^2) + (a_{11}a_{22} - a_{12}^2) \\
p_2 &= -(a_{00} + a_{11} + a_{22}) = -\,\mathrm{Trace}(A)
\end{aligned}
\tag{4}
$$

The polynomial has only real-valued roots. Closed-form equations exist for the roots. When the roots are distinct, the roots are generally not rational numbers. If A has rational components, the roots are generally not rational numbers. However, when at least one root is repeated, the roots are rational.

If A and b have rational components and c is rational, obtaining an exact classification of the solution set for the quadratic equation is possible but requires formulations that ensure various expressions are rational valued.

2.2 Descartes' Rule of Signs

Descartes' rule of signs [1] applies to polynomials $p(\lambda)$ with real-valued coefficients. If the nonzero real-valued coefficients of $p(\lambda)$ are ordered by descending exponent, then the number of positive roots is either the number of sign changes of the ordered coefficients or the number of sign changes minus an even number. A root of multiplicity m is counted as m roots. When the number of positive roots is less than the number of sign changes, the polynomial has pairs of non-real roots that are complex conjugates. The number of negative roots of $p(\lambda)$ is the number of positive roots of $p(-\lambda)$.

The characteristic polynomial $p(\lambda)$ for a real-valued symmetric matrix A has only real-valued roots. If A is $n \times n$, the k distinct roots are r_i for $0 \le i < k$, each having multiplicity m_i. The sum of the multiplicities is n. Descartes' rule of signs produces the exact number of positive roots, say, n_+, and the exact number of negative roots, say, n_-. Let n_0 be the number of zero roots; then $n_+ + n_- + n_0 = n$. Listing 1 contains pseudocode for computing the numbers of roots of $p(\lambda)$.

Listing 1. Pseudocode for Descartes' rule of signs to count the number of positive roots, the number of negative roots and the number of zero roots for the characteristic polynomial $p(\lambda) = \det(A - \lambda I)$.

```
void ComputeRootSigns(Polynomial<Rational> p, int& numPositive, int& numNegative, int& numZero)
{
    // Assert n >= 1 and p[n] is not zero.
    int n = Degree(p);

    // Compute the signs of the coefficients of p(lambda). The values are in {-1,0,1}.
    int signs[n + 1];
    for (int i = 0; i <= n; ++i)
    {
        signs[i] = Sign(p[i]);
    }

    // Compute the number of positive roots of p(lambda).
    int currentSign = signs[n];
    numPositive = 0;
    for (int i = n-1; i >= 0; --i)
    {
        if (signs[i] == -currentSign)
        {
            currentSign = -1;
            ++numPositive;
        }
    }

    // Compute the signs of the coefficients of p(-lambda).
    for (int i = 0; i <= n; i += 2)
    {
        signs[i] -= -signs[i];
    }

    // Compute the number of negative roots of p(lambda).
    currentSign = signs[n];
    numNegative = 0;
    for (int i = n-1; i >= 0; --i)
    {
        if (sign[i] == -currentSign)
        {
```

```
            currentSign = −1;
            ++numNegative;
        }
    }

    // Compute the number of zero roots of p(lambda).
    numZero = n − numPositive − numNegative;
}
```

3   Classification of the Solution Set of the Quadratic Equation

3.1   All Nonzero Eigenvalues

The matrix A is invertible because all eigenvalues are not zero. Define $\boldsymbol{u} = -A^{-1}\boldsymbol{b}/2$. Equation (1) factors into $(\boldsymbol{x} - \boldsymbol{u})^{\mathsf{T}} A (\boldsymbol{x} - \boldsymbol{u}) = \boldsymbol{u}^{\mathsf{T}} A \boldsymbol{u} - c$. Using the eigendecomposition $A = R \Lambda R^{\mathsf{T}}$ and defining $\boldsymbol{y} = R^{\mathsf{T}}(\boldsymbol{x} - \boldsymbol{u})$, the equation is further modified to

$$\lambda_0 y_0^2 + \lambda_1 y_1^2 + \lambda_2 y_2^2 = \boldsymbol{y}^{\mathsf{T}} \Lambda \boldsymbol{y} = \frac{1}{4} \boldsymbol{b}^{\mathsf{T}} A^{-1} \boldsymbol{b} - c = r \tag{5}$$

where the last equality defines r. Observe that r is rational whenever A and $\boldsymbol{b}$ have rational components and c is rational. Table 1 shows the signs of the relevant values and the classifications determined by them.

Table 1. Classifications when all eigenvalues are not zero.

λ_0	λ_1	λ_2	r	classification
$+$	$+$	$+$	$+$	ellipsoid
$-$	$+$	$+$	$+$	hyperboloid of one sheet
$-$	$-$	$+$	$+$	hyperboloid of two sheets
$-$	$-$	$-$	$+$	no solution
$+$	$+$	$+$	0	point
$-$	$+$	$+$	0	elliptic cone
$-$	$-$	$+$	0	elliptic cone
$-$	$-$	$-$	0	point
$+$	$+$	$+$	$-$	no solution
$-$	$+$	$+$	$-$	hyperboloid of two sheets
$-$	$-$	$+$	$-$	hyperboloid of one sheet
$-$	$-$	$-$	$-$	ellipsoid

Listing 2 contains pseudocode for the classification when all eigenvalues are not zero.

Listing 2. Pseudocode for the classification of the solution set to Equation (1) when all eigenvalues are not zero.

```
Classification AllNonzero(Matrix3x3<Rational> A, Vector3<Rational> b, Rational c, int numPositive)
{
    Rational r = Dot(b, Inverse(A) * b) / 4 − c;
    if (numPositive == 3)
    {
        return (r > 0 ? ELLIPSOID : (r < 0 ? NO_SOLUTION : POINT));
    }
    else if (numPositive == 2)
    {
        return (r > 0 ? HYPERBOLOID_ONE_SHEET : (r < 0 ? HYPERBOLOID_TWO_SHEETS : ELLIPTIC_CONE));
    }
    else if (numPositive == 1)
    {
        return (r > 0 ? HYPERBOLOID_TWO_SHEETS : (r < 0 ? HYPERBOLOID_ONE_SHEET : ELLIPTIC_CONE));
    }
    else
    {
        return (r != 0 ? ELLIPTIC_CONE : POINT);
    }
}
```

3.2 Two Nonzero Eigenvalues

Consider when two eigenvalues are not 0 but the third eigenvalue is 0. The rank of A is 2 because it has an eigenvalue 0 of multiplicity 1. Let the 2 linearly independent rows of A be $\boldsymbol{\rho}_1$ and $\boldsymbol{\rho}_2$. An eigenvector for eigenvalue 0 is $\boldsymbol{w}_0 = \boldsymbol{\rho}_1 \times \boldsymbol{\rho}_2$ and is not necessarily unit length. If A has rational components, then $\boldsymbol{w}_0$ has rational components. Choose $\boldsymbol{w}_1 = \boldsymbol{p}_1$ which has rational components and is perpendicular to $\boldsymbol{w}_0$. Define $\boldsymbol{w}_2 = \boldsymbol{w}_0 \times \boldsymbol{w}_1$ which also has rational components. The set $\{\boldsymbol{w}_0, \boldsymbol{w}_1, \boldsymbol{w}_2\}$ is an orthogonal set but not an orthonormal set; that is, the vectors are mutually perpendicular but they are not necessarily unit length.

Define the matrix $W = [\boldsymbol{w}_0 \, \boldsymbol{w}_1 \, \boldsymbol{w}_2]$ whose columns are the specified vectors. Define $\boldsymbol{x} = W\boldsymbol{y}$ and $\boldsymbol{d} = W^{\mathsf{T}}\boldsymbol{b}$. Equation (1) becomes

$$
\begin{aligned}
0 &= \boldsymbol{y}^{\mathsf{T}} W^{\mathsf{T}} A W \boldsymbol{y} + \boldsymbol{d}^{\mathsf{T}} \boldsymbol{y} + c \\[2mm]
&= \left[\begin{array}{c|cc} y_0 & y_1 & y_2 \end{array}\right]
\left[\begin{array}{c|cc} 0 & 0 & 0 \\ \hline 0 & \boldsymbol{w}_1^{\mathsf{T}} A \boldsymbol{w}_1 & \boldsymbol{w}_1^{\mathsf{T}} A \boldsymbol{w}_2 \\ 0 & \boldsymbol{w}_2^{\mathsf{T}} A \boldsymbol{w}_1 & \boldsymbol{w}_2^{\mathsf{T}} A \boldsymbol{w}_2 \end{array}\right]
\left[\begin{array}{c} y_0 \\ \hline y_1 \\ y_2 \end{array}\right]
+ \left[\begin{array}{c|cc} d_0 & d_1 & d_2 \end{array}\right]
\left[\begin{array}{c} y_0 \\ \hline y_1 \\ y_2 \end{array}\right] + c
\end{aligned}
\tag{6}
$$

Observe that d_0 is rational.

Define the quantities

$$
E = \left[\begin{array}{cc} \boldsymbol{w}_1^{\mathsf{T}} A \boldsymbol{w}_1 & \boldsymbol{w}_1^{\mathsf{T}} A \boldsymbol{w}_2 \\ \boldsymbol{w}_2^{\mathsf{T}} A \boldsymbol{w}_1 & \boldsymbol{w}_2^{\mathsf{T}} A \boldsymbol{w}_2 \end{array}\right], \quad
\boldsymbol{f} = \left[\begin{array}{c} d_1 \\ d_2 \end{array}\right], \quad
\boldsymbol{\eta} = \left[\begin{array}{c} y_1 \\ y_2 \end{array}\right]
\tag{7}
$$

Equation (6) becomes

$$
d_0 y_0 + \boldsymbol{\eta}^{\mathsf{T}} E \boldsymbol{\eta} + \boldsymbol{f}^{\mathsf{T}} \boldsymbol{\eta} + c = 0
\tag{8}
$$

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The matrices A and $W^\mathsf{T} A W$ are congruent [2], so by Sylvester's law of inertia they have the same numbers of positive, negative and zero eigenvalues; therefore, E has 2 nonzero eigenvalues. Its eigendecomposition is $E = Q \operatorname{Diagonal}(\mu_1, \mu_2) Q^\mathsf{T}$ where Q is a rotation matrix.

Define $\boldsymbol{k} = -E^{-1}\boldsymbol{f}/2$, $\boldsymbol{z} = Q^\mathsf{T}(\boldsymbol{\eta} - \boldsymbol{k})$ with $\boldsymbol{z}^\mathsf{T} = [z_1\ z_2]$, and $z_0 = y_0$. Equation (6) becomes

$$d_0 z_0 + \mu_1 z_1^2 + \mu_2 z_2^2 = \frac{1}{4}\boldsymbol{f}^\mathsf{T} E^{-1} \boldsymbol{f} - c = r \tag{9}$$

where the last equality defines r, a rational number because E and $\boldsymbol{f}$ have rational components and c is a rational number.

Table 2 shows the signs of the relevant values and the classifications determined by them.

Table 2. Classifications for two positive eigenvalues, one positive and one negative eigenvalue, or two negative eigenvalues. Cells without signs indicate the value is irrelevant. Cells with $\pm$ indicate the value is not zero.

two positive				one positive and one negative				two negative		
d_0	r	classification		d_0	r	classification		d_0	r	classification
$\pm$		elliptic paraboloid		$\pm$		hyperbolic paraboloid		$\pm$		elliptic paraboloid
0	$+$	elliptic cylinder		0	$\pm$	hyperbolic cylinder		0	$-$	elliptic cylinder
0	0	line		0		two planes		0	0	line
0	$-$	no solution						0	$+$	no solution

Listing 3 contains pseudocode for the classification.

Listing 3. Pseudocode for the classification of the solution set to Equation (1) when there are two nonzero eigenvalues.

```
void ComputeOrthogonalSetTwoNonzero(Matrix3x3<Rational> A,
    Vector3<Rational>& w0, Vector3<Rational>& w1, Vector3<Rational>& w2)
{
    Vector3<Rational> vzero = { 0, 0, 0 };
    w1 = { A(0, 0), A(0, 1), A(0, 2) };
    if (w1 != vzero)
    {
        w2 = { A(1, 0), A(1, 1), A(1, 2) };
        if (Cross(w1, w2) == vzero)
        {
            w2 = { A(2, 0), A(2, 1), A(2, 2) };
        }
    }
    else
    {
        w1 = { A(1, 0), A(1, 1), A(1, 2) };
        w2 = { A(2, 0), A(2, 1), A(2, 2) };
    }

    w0 = Cross(w1, w2);
    w2 = Cross(w0, w1);
}
```

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```
Classification TwoNonzero(Matrix3x3<Rational> A, Vector3<Rational> b, Rational c,
    int numPositive, int numNegative)
{
    Vector3<Rational> w0, w1, w2;
    ComputeOrthogonalSetTwoNonzero(A, w0, w1, w2);
    Rational d0 = Dot(w0, b);
    if (d0 != 0)
    {
        return (numPositive == numNegative ? HYPERBOLIC_PARABOLOID : ELLIPTIC_PARABOLOID);
    }

    Vector3<Rational> Aw1 = A * w1, Aw2 = A * w2;
    Matrix2x2<Rational> E = {{ Dot(w1, Aw1), Dot(w1, Aw2) }, { Dot(w1, Aw2), Dot(w2, Aw2) }};
    Vector2<Rational> f = { Dot(w1, b), Dot(w2, b) };
    Rational r = Dot(f, Inverse(E) * f) / 4 - c;

    if (numPositive == 2)
    {
        return (r > 0 ? ELLIPTIC_CYLINDER : (r < 0 ? NO_SOLUTION : LINE));
    }
    else if (numNegative == 2)
    {
        return (r != 0 ? HYPERBOLIC_CYLINDER : TWO_PLANES);
    }
    else  // numPositive = numNegative = 1
    {
        return (r < 0 ? ELLIPTIC_CYLINDER : (r > 0 ? NO_SOLUTION : LINE));
    }
}
```

3.3 One Nonzero Eigenvalue

Consider when one eigenvalue is not 0 but the other two eigenvalues are 0. The rank of A is 1 because it has an eigenvalue 0 of multiplicity 2. Let the nonzero row be $\boldsymbol{w}_2$. It is an eigenvector for the nonzero eigenvalue; it has rational components but is not necessarily unit length. Choose a rational-valued vector $\boldsymbol{w}_0$ that is perpendicular to $\boldsymbol{w}_2$ and define $\boldsymbol{w}_1 = \boldsymbol{w}_2 \times \boldsymbol{w}_0$. The vectors $\boldsymbol{w}_0$ and $\boldsymbol{w}_1$ are linearly independent eigenvectors for eigenvalue 0.

Define the matrix $W = [\boldsymbol{w}_0 \, \boldsymbol{w}_1 \, \boldsymbol{w}_2]$ whose columns are the specified vectors. Define $\boldsymbol{x} = W\boldsymbol{y}$ and $\boldsymbol{d} = W^\mathsf{T}\boldsymbol{b}$. Equation (1) becomes

$$
\begin{aligned}
0 &= \boldsymbol{y}^\mathsf{T} W^\mathsf{T} A W \boldsymbol{y} + \boldsymbol{d}^\mathsf{T} \boldsymbol{y} + c \\
&= \begin{bmatrix} y_0 & y_1 & y_2 \end{bmatrix} \left[\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & \boldsymbol{w}_2^\mathsf{T} A \boldsymbol{w}_2 \end{array} \right] \begin{bmatrix} y_0 \\ y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} d_0 & d_1 & d_2 \end{bmatrix} \begin{bmatrix} y_0 \\ y_1 \\ y_2 \end{bmatrix} + c
\end{aligned}
\tag{10}
$$

Observe that d_0 and d_1 are rational numbers.

Equation (10) can be factored to

$$
d_0 z_0 + d_1 z_2 + \mu_2 z_2^2 = \frac{1}{4} f E^{-1} f - c = r
\tag{11}
$$

where $\mu_2 = E = \boldsymbol{w}_2^\mathsf{T} A \boldsymbol{w}_2$, $f = d_2$, $z_0 = y_0$, $z_1 = y_1$ and $z_2 = y_2 + d_2/(2e_2)$. The last equality defines r, a rational number.

Table 3 shows the signs of the relevant values and the classifications determined by them.

Table 3. Classifications for exactly two zero eigenvalues. Cells without signs indicate the value is irrelevant. Cells with $\pm$ indicate the value is not zero.

one positive					one negative			
d_0	d_1	r	classification		d_0	d_1	r	classification
$\pm$			parabolic cylinder		$\pm$			parabolic cylinder
	$\pm$		parabolic cylinder			$\pm$		parabolic cylinder
0	0	+	two planes		0	0	$-$	two planes
0	0	0	one plane		0	0	0	one plane
0	0	$-$	no solution		0	0	+	no solution

Listing 4 contains pseudocode for the classification when there are exactly two zero eigenvalues.

Listing 4. Pseudocode for the classification of the solution set to Equation (1) when there are two zero eigenvalues.

```
void ComputeOrthogonalSetOneNonzero(Matrix3x3<Rational> A,
    Vector3<Rational>& w0, Vector3<Rational>& w1, Vector3<Rational>& w2)
{
    Vector3<Rational> vzero = { 0, 0, 0 };
    w2 = { A(0, 0), A(0, 1), A(0, 2) }
    if (w2 == vzero)
    {
        w2 = { A(1, 0), A(1, 1), A(1, 2) };
        if (w2 == vzero)
        {
            w2 = { A(2, 0), A(2, 1), A(2, 2) };
        }
    }

    if (abs(w2[0]) > abs(w2[1]))
    {
        w0 = { -w2[2], 0, +w2[0] };
    }
    else
    {
        w0 = { 0, +w2[2], -w2[1] };
    }
    w1 = Cross(w2, w0);
}

Classification OneNonzero(Matrix3x3<Rational> A, Vector3<Rational> b, Rational c, int numPositive)
{
    Vector3<Rational> w0, w1, w2;
    ComputeOrthogonalSetOneNonzero(w0, w1, w2);
    Rational d0 = Dot(w0, b);
    Rational d1 = Dot(w1, b);
    if (d0 != 0 || d1 != 0)
    {
        return PARABOLIC_CYLINDER;
    }

    Rational e2 = Dot(w2, A * w2);
    Rational d2 = Dot(w2, b);
    Rational r = d2 * d2 / (4 * e2) - c;
```

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```
    if ( numPositive == 1 )    // numNegative = 0
    {
        return ( r > 0 ? TWO_PLANES: ( r < 0 ? NO_SOLUTION : ONE_PLANE ));
    }
    else   // numPositive = 0, numNegative = 1
    {
        return ( r < 0 ? TWO_PLANES: ( r > 0 ? NO_SOLUTION : ONE_PLANE ));
    }
}
```

3.4   All Zero Eigenvalues

The quadratic equation degenerates to the linear equation $b_0 x_0 + b_1 x_1 + b_2 x_2 + c = 0$. Table 4 shows the signs of the relevant values and the classifications determined by them.

Table 4. Classifications all eigenvalues are zero. Cells without signs indicate the value is irrelevant. Cells with $\pm$ indicate the value is not zero.

b_0	b_1	b_2	c	classification
$\pm$				plane
	$\pm$			plane
		$\pm$		plane
0	0	0	$\pm$	no solution
0	0	0	0	all points are solutions

Listing 5 contains pseudocode for the classification.

Listing 5.   Pseudocode for the classification when all eigenvalues are 0.

```
Classification AllZero ( Vector3<Rational> b, Rational c )
{
    Vector3<Rational> vzero = { 0, 0, 0 };
    if ( b != vzero )
    {
        return PLANE;
    }
    else
    {
        return c == 0 ? ENTIRE_SPACE : NO_SOLUTION;
    }
}
```

References

[1] Wikipedia. Descartes' rule of signs.
    https://en.wikipedia.org/wiki/Descartes'_rule_of_signs.
    accessed August 28, 2022.

[2] Wikipedia. Matrix congruence.
https://en.wikipedia.org/wiki/Matrix_congruence.
accessed September 5, 2022.