

Bicubic Bézier Exact Interpolation

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1 Introduction

Given a 4×4 grid of points, a bicubic Bézier surface patch may be constructed to pass through the 16 points, thus leading to an exact interpolation of the points.

2 The Algorithm

Let the data points be $\mathbf{P}_{ij}$ for $0 \leq i \leq 3$ and $0 \leq j \leq 3$. A bicubic Bézier patch $\mathbf{X}(s, t)$ is of the form shown below where the coefficient of $(1-s)^{3-i}s^i$ is the combinatorial symbol for 3 choose i and has value $3!/((3-i)!i!)$. Similarly, the coefficient of $(1-t)^{3-j}t^j$ has value $3!/((3-j)!j!)$. The domain is $(s, t) \in [0, 1]^2$.

$$\begin{aligned}
 \mathbf{X}(s, t) &= \sum_{i=0}^3 \sum_{j=0}^3 \binom{3}{i} (1-s)^{3-i} s^i \binom{3}{j} (1-t)^{3-j} t^j \mathbf{C}_{ij} \\
 &= \begin{aligned} &(1-s)^3 \quad [\quad (1-t)^3 \mathbf{C}_{00} \quad + \quad 3(1-t)^2 t \mathbf{C}_{01} \quad + \quad 3(1-t) t^2 \mathbf{C}_{02} \quad + \quad t^3 \mathbf{C}_{03} \quad] \\ &+ \quad 3(1-s)^2 s \quad [\quad (1-t)^3 \mathbf{C}_{10} \quad + \quad 3(1-t)^2 t \mathbf{C}_{11} \quad + \quad 3(1-t) t^2 \mathbf{C}_{12} \quad + \quad t^3 \mathbf{C}_{13} \quad] \\ &+ \quad 3(1-s) s^2 \quad [\quad (1-t)^3 \mathbf{C}_{20} \quad + \quad 3(1-t)^2 t \mathbf{C}_{21} \quad + \quad 3(1-t) t^2 \mathbf{C}_{22} \quad + \quad t^3 \mathbf{C}_{23} \quad] \\ &+ \quad s^3 \quad [\quad (1-t)^3 \mathbf{C}_{30} \quad + \quad 3(1-t)^2 t \mathbf{C}_{31} \quad + \quad 3(1-t) t^2 \mathbf{C}_{32} \quad + \quad t^3 \mathbf{C}_{33} \quad] \end{aligned} \tag{1}
 \end{aligned}$$

We want to choose the $\mathbf{C}_{ij}$ so that $\mathbf{X}(i/3, j/3) = \mathbf{P}_{ij}$ for $0 \leq i \leq 3$ and $0 \leq j \leq 3$. This gives us a linear system of 16 vector-valued equations in 16 unknowns, The matrix of coefficients is invertible, leading to the

solution

$$\begin{aligned}
\mathbf{C}_{00} &= \mathbf{P}_{00} \\
\mathbf{C}_{01} &= (-5\mathbf{P}_{00} + 18\mathbf{P}_{01} - 9\mathbf{P}_{02} + 2\mathbf{P}_{03})/6 \\
\mathbf{C}_{02} &= (2\mathbf{P}_{00} - 9\mathbf{P}_{01} + 18\mathbf{P}_{02} - 5\mathbf{P}_{03})/6 \\
\mathbf{C}_{03} &= \mathbf{P}_{03} \\
\mathbf{C}_{10} &= (-5\mathbf{P}_{00} + 18\mathbf{P}_{10} - 9\mathbf{P}_{20} + 2\mathbf{P}_{30})/6 \\
\mathbf{C}_{11} &= (25\mathbf{P}_{00} - 90\mathbf{P}_{01} + 45\mathbf{P}_{02} - 10\mathbf{P}_{03} \\
&\quad - 90\mathbf{P}_{10} + 324\mathbf{P}_{11} - 162\mathbf{P}_{12} + 36\mathbf{P}_{13} \\
&\quad + 45\mathbf{P}_{20} - 162\mathbf{P}_{21} + 81\mathbf{P}_{22} - 18\mathbf{P}_{23} \\
&\quad - 10\mathbf{P}_{30} + 36\mathbf{P}_{31} - 18\mathbf{P}_{32} + 4\mathbf{P}_{33})/36 \\
\mathbf{C}_{12} &= (-10\mathbf{P}_{00} + 45\mathbf{P}_{01} - 90\mathbf{P}_{02} + 25\mathbf{P}_{03} \\
&\quad + 36\mathbf{P}_{10} - 162\mathbf{P}_{11} + 324\mathbf{P}_{12} - 90\mathbf{P}_{13} \\
&\quad - 18\mathbf{P}_{20} + 81\mathbf{P}_{21} - 162\mathbf{P}_{22} + 45\mathbf{P}_{23} \\
&\quad + 4\mathbf{P}_{30} - 18\mathbf{P}_{31} + 36\mathbf{P}_{32} - 10\mathbf{P}_{33})/36 \\
\mathbf{C}_{13} &= (-5\mathbf{P}_{03} + 18\mathbf{P}_{13} - 9\mathbf{P}_{23} + 2\mathbf{P}_{33})/6 \\
\mathbf{C}_{20} &= (2\mathbf{P}_{00} - 9\mathbf{P}_{10} + 18\mathbf{P}_{20} - 5\mathbf{P}_{30})/6 \\
\mathbf{C}_{21} &= (-10\mathbf{P}_{00} + 36\mathbf{P}_{01} - 18\mathbf{P}_{02} + 4\mathbf{P}_{03} \\
&\quad + 45\mathbf{P}_{10} - 162\mathbf{P}_{11} + 81\mathbf{P}_{12} - 18\mathbf{P}_{13} \\
&\quad - 90\mathbf{P}_{20} + 324\mathbf{P}_{21} - 162\mathbf{P}_{22} + 36\mathbf{P}_{23} \\
&\quad + 25\mathbf{P}_{30} - 90\mathbf{P}_{31} + 45\mathbf{P}_{32} - 10\mathbf{P}_{33})/36 \\
\mathbf{C}_{22} &= (4\mathbf{P}_{00} - 18\mathbf{P}_{01} + 36\mathbf{P}_{02} - 10\mathbf{P}_{03} \\
&\quad - 18\mathbf{P}_{10} + 81\mathbf{P}_{11} - 162\mathbf{P}_{12} + 45\mathbf{P}_{13} \\
&\quad + 36\mathbf{P}_{20} - 162\mathbf{P}_{21} + 324\mathbf{P}_{22} - 90\mathbf{P}_{23} \\
&\quad - 10\mathbf{P}_{30} + 45\mathbf{P}_{31} - 90\mathbf{P}_{32} + 25\mathbf{P}_{33})/36 \\
\mathbf{C}_{23} &= (2\mathbf{P}_{03} - 9\mathbf{P}_{13} + 18\mathbf{P}_{23} - 5\mathbf{P}_{33})/6 \\
\mathbf{C}_{30} &= \mathbf{P}_{30} \\
\mathbf{C}_{31} &= (-5\mathbf{P}_{30} + 18\mathbf{P}_{31} - 9\mathbf{P}_{32} + 2\mathbf{P}_{33})/6 \\
\mathbf{C}_{32} &= (2\mathbf{P}_{30} - 9\mathbf{P}_{31} + 18\mathbf{P}_{32} - 5\mathbf{P}_{33})/6 \\
\mathbf{C}_{33} &= \mathbf{P}_{33}
\end{aligned} \tag{2}$$