Polysolids and Boolean Operations

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1 Introduction

This document is based on notes from Professor Hugh Maynard and Professor Lucio Tavernini in the Computer Science Department at the University of Texas at San Antonio. It is also based on my recollection of the ideas when I attended as a visitor a seminar by Professor Maynard.

The framework for polysolids was developed by them in the early 1980s, but they never published their work. At that time, the seminal book *Computational Geometry: An Introduction* by Franco P. Preparata and Michael Ian Shamos was published. The Maynard–Tavernini ideas are important for Boolean operations in constructive planar geometry (2D) and in constructive solid geometry (3D). Moreover, the ideas easily extend to higher dimensions and reduce to the 2D problem by recursion in dimension. Boolean operations on polysolids require no special-case handling as the framework is concise and powerful.

The original title of this document was *Polysolids and Boolean Operations*, but I renamed it to emphasize the focus on Boolean operations. I also have taken the liberty to present the ideas in a modified form to emphasize the design and implementation of the Boolean operations. During the process of revisiting the material I have restricted the definition of polysolids to those having a bounded component, which is all that is needed for Boolean operations in standard applications. Any interpretations or misinterpretations of the Maynard–Tavernini concepts are my own without their input.

2 Topological Concepts

The discussion is restricted to $\mathbb{R}^n$, the set of n-tuples with $n \geq 1$ and whose components are real numbers. For $n = 1$, $\mathbb{R}$ is typically used to denote the set of real numbers rather than $\mathbb{R}^1$, although the latter still refers to the set of real numbers.

The set $\mathbb{R}^n$ is a *metric space* with a distance function $d(x, y) = |x - y|$ for $x, y \in \mathbb{R}^n$, which is the square root of the sum of squared differences of the components of x and y. The distance function satisfies $d(x, y) = 0$ if and only if $x = y$. It is symmetric because $d(x, y) = d(y, x)$. Finally, the triangle inequality is satisfied, $d(x, y) + d(y, z) \geq d(x, z)$ for all $x, y, z \in \mathbb{R}^n$.

A *ball* in $\mathbb{R}^n$ has center c and radius $r > 0$ and is defined by $B(c, r) = \{x \in S : |x - c| < r\}$. A set $S \subset \mathbb{R}^n$ is *open* if every point $c \in S$ has a ball of center c and radius $r > 0$ for which $B(c, r) \subset S$. The union of open sets is an open set and the intersection of a finite number of open sets is an open set. The empty set $\emptyset$ is considered to be an open set and $\mathbb{R}^n$ itself is considered to be an open set. The collection of open sets in $\mathbb{R}^n$ is said to be a *topology*.

Open intervals are sets of the form $(a, b) = \{x \in \mathbb{R} : a < x < b\}$, where $-\infty \leq a < b \leq +\infty$. The endpoints a and b are not elements of the open interval. The intervals $(-\infty, b)$ for $b < +\infty$ and $(a, +\infty)$ for $-\infty < a$ are referred to as *semiinfinite intervals* and are unbounded sets. The *infinite interval* is $(-\infty, +\infty) = \mathbb{R}$. Open intervals are open sets. For example, (a, b) for finite a and b is open because given any $c \in (a, b)$, choose radius $r = \min\{|c - a|, |c - b|\}$. It is the case that $B(c, r) \subset (a, b)$.

The *complement* of a set $S \subset \mathbb{R}^n$ is the set difference $\mathbb{R}^n \setminus S = \{x \in \mathbb{R} : x \notin S\}$. This is denoted $\mathrm{Complement}(S)$.

A set $S \subset \mathbb{R}^n$ is a *closed set* when its complement $\mathbb{R}^n \setminus S$ is an open set.

Closed intervals are sets of the form $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$, where $-\infty < a < b < +\infty$. The endpoints

a and b are elements of the closed interval. The complement of $[a, b]$ is the set $\mathbb{R} \setminus [a, b] = (-\infty, a) \cup (b, +\infty)$, the right-hand side the union of two semiinfinite open intervals which makes it an open set. Because the complement of $[a, b]$ is an open set, $[a, b]$ is a closed set.

Given a set $S \subset \mathbb{R}^n$, the point $x \in \mathbb{R}^n$ is a *limit point* of S when for each ball $B(x, r)$, there exists $y \in S \cap B(x, r)$. For example, consider $S = [a, b] \subset \mathbb{R}$. The endpoint a is limit point of S because $B(a, r)$ for any $r > 0$ contains a point $y \in [a, b]$. One such choice for y when $r \leq b - a$ is $y = a + r/2$. The limit as r approaches 0 is a, which explains the use of the term limit point. An equivalent definition for a closed set is that is a set that contains all its limit points.

Given any $S \subset \mathbb{R}^n$, the *closure* of S is the union of S and its limit points. This is denoted Closure(S), and the closure of a set is necessarily a closed set.

Given a set $S \subset \mathbb{R}^n$, the *interior* of S is the union of all its open subsets. This is denoted Interior(S), and the interior of a set is necessarily an open set.

3 Polysolids

A *polysolid* is a generalization of the concept of polygonal regions in 2D and polyhedral regions in 3D. The ideas apply in dimensions larger than three, reduced to the 2D problem using recursion in dimension. The intersection of a hyperplane with an n-dimensional polysolid is an $(n-1)$-dimensional polysolid within that hyperplane. The base case of the recursive definition is 1D, where the motivation is provided first in the next section. One may actually recurse to 0-dimensional polysolids, which are simply points, but this is not useful in practical applications. This document contains a discussion of Boolean operations of polysolids in 2D and in 3D.

3.1 Polysolids in $\mathbb{R}^1$

Let $L = \{\ell_1, \ldots, \ell_m\} \subset \mathbb{R}^1$ be a finite set of real numbers. The complement of L is

$$\mathbb{R}^1 \setminus L = \bigcup_{i=0}^{m} U_i \tag{1}$$

where the U_i are the open intervals $U_0 = (-\infty, \ell_1)$, $U_i = (\ell_i, \ell_{i+1})$ for $1 \leq i \leq m-1$ and $U_m = (\ell_m, +\infty)$. A graph associated with L is $G_L = (V_L, E_L)$, where the graph vertices are $V_L = \{U_i\}_{i=0}^{m}$ and the graph edges are $E_L = \{(U_i, U_{i+1}) : 0 \leq i \leq m-1\}$.

G_L is a connected 2-colorable graph with colors 0 and 1. Define $c_L : V_L \to \{0, 1\}$ to be the unique 2-coloring of G_L such that $c_L(U_0) = 1$. For example, if $L = \{\ell_0\}$, then $U_0 = (-\infty, \ell_0)$ and $U_1 = (\ell_0, +\infty)$ with $c_L(U_0) = 1$ and $c_L(U_1) = 0$. If $L = \{\ell_0, \ell_1\}$, then $U_0 = (-\infty, \ell_0)$, $U_1 = (\ell_0, \ell_1)$ and $U_2 = (\ell_1, +\infty)$ with $c_L(U_0) = 1$, $c_L(U_1) = 0$ and $c_L(U_2) = 1$. Finally, if $L = \{\ell_0, \ell_1, \ell_2\}$, then $U_0 = (-\infty, \ell_0)$, $U_1 = (\ell_0, \ell_1)$, $U_2 = (\ell_1, \ell_2)$ and $U_3 = (\ell_2, +\infty)$ with $c_L(U_0) = c_L(U_2) = 1$ and $c_L(U_1) = c_L(U_3) = 0$.

L defines two complementary *polysolids*,

$$P_\alpha(L) = \{U \in V_L : c_L(U) = \alpha\} \tag{2}$$

for $\alpha \in \{0, 1\}$. For example, if $L = \{\ell_0, \ell_1, \ell_2\}$, then $P_0(L) = (-\infty, \ell_0) \cup (\ell_1, \ell_2)$ and $P_1(L) = (\ell_0, \ell_1) \cup (\ell_2, +\infty)$. Both $P_0(L)$ and $P_1(L)$ are unbounded sets. If $L = \{\ell_0, \ell_1\}$, then $P_0(L) = (-\infty, \ell_0) \cup (\ell_1, +\infty)$

and $P_1(L) = (\ell_0, \ell_1)$. $P_0(L)$ is unbounded and $P_1(L)$ is bounded. Generally, if L has an odd number of points, then both polysolids are unbounded. If L has an even number of points, then $P_0(L)$ is unbounded and $P_1(L)$ is bounded. In this case, $P_0(L)$ is said to be *cobounded*; its complement $\mathbb{R}^1 \setminus P_0(L)$ is a bounded set.

The set of 1-dimensional polysolids in $\mathbb{R}^1$ is defined by

$$\mathcal{P}^1(\mathbb{R}^1) = \{P_\alpha(L) : L \subset \mathbb{R}^1 \text{ is finite and } \alpha \in \{0, 1\}\} \tag{3}$$

The definition extends to lines embedded in spaces of dimension $d > 1$. If $H \subset \mathbb{R}^d$ is a 1-flat, then $\mathcal{P}^1(H)$ is defined to be the image of $\mathcal{P}^1(\mathbb{R}^1)$ under any affine transformation of $\mathbb{R}^1$ onto H. The set of all 1-dimensional polysolids in $\mathbb{R}^d$ is

$$\mathcal{P}^1(\mathbb{R}^d) = \bigcup\{\mathcal{P}^1(H) : H \text{ is a 1-flat in } \mathbb{R}^d\} \tag{4}$$

An important subset of $\mathcal{P}^1(\mathbb{R}^1)$ is the collection of pairs of bounded-cobounded polysolids, denoted $\mathcal{P}^1_b(\mathbb{R}^1)$. As noted previously, these polysolids are generated by finite sets L with an even number of points. The set of bounded-cobounded polysolids corresponding to a 1-flat H in $\mathbb{R}^d$ is denoted $\mathcal{P}^1_b(H)$ and the set of bounded-cobounded polysolids in $\mathbb{R}^d$ is denoted $\mathcal{P}^1_b(\mathbb{R}^d)$.

3.2 Polysolids in $\mathbb{R}^2$

Let $L \subset \mathcal{P}^1(\mathbb{R}^2)$ have a finite number of elements, each element a 1-dimensional polysolid. The set

$$\mathbb{R}^2 \setminus \text{Closure}(L) = \mathbb{R}^2 \setminus \bigcup_{e \in L} \text{Closure}(e) \tag{5}$$

is an open set that is a disjoint union of a finite number of open connected sets, $V_L = \{U_i\}_{i=0}^m$.

The set L is said to be *decomposing in* $\mathbb{R}^2$ if and only if V_L is composed of open sets U, each satisfying the topological condition

$$\text{Interior}(\text{Closure}(U)) = U \tag{6}$$

An open set U satisfying this condition is said to be *regular*. The use of decomposing finite sets avoids dangling line segments. For example, the 1-dimensional bounded polysolids $(x, 0)$, $(x, 1)$, $(0, y)$ and $(1, y)$ with $0 \le x \le 1$ and $0 \le y \le 1$ form a set L that is decomposing. The set $\cup_{e \in L}\text{Closure}(e)$ is the perimeter of the unit square $[0, 1]^2$. The two regions inside and outside the square are regular sets; see Figure 1 (a). However if L additionally contains the polysolid (x, x) for $0 \le x \le 1/2$, then L is not decomposing. The diagonal edge is a dangling line segment for the polysolid; see Figure 1 (b).

Figure 1. Decomposing and nondecomposing sets. In each of (a) and(b), the set L is shown on the left and the set $\mathbb{R}^2 \setminus \text{Closure}(L)$ is shown on the right.

(a) L is decomposing. (b) L is not decomposing.

A graph associated with L is $G_L = (V_L, E_L)$, where $(U_i, U_j) \in E_L$ if and only if $U_i \cup U_j$ is not a regular open set. Intuitively, U_i and U_j are separated by a 1-dimensional piecewise linear curve.

The set L is said to be *generating in* $\mathbb{R}^2$ if and only if it is decomposing and G_L is 2-colorable. For example, the four polysolids mentioned previously whose union of closures forms the unit square is generating; see Figure 2 (a). If L additionally contains the polysolid (x, x) for $0 \le x \le 1$, then L is decomposing but not generating because the graph G_L is not 2-colorable; see Figure 2 (b).

Figure 2. Generating and nongenerating sets. In each of (a) and(b), the set L is shown on the left and the set $\mathbb{R}^2 \setminus \mathrm{Closure}(L)$ is shown on the right.

(a) L is generating. (b) L is decomposing but not generating.

A generating set L defines two complementary *polysolids*,

$$P_\alpha(L) = \{U \in V_L : c(U) = \alpha\} \tag{7}$$

where $c : V_L \to \{0, 1\}$ is a 2-coloring of G_L. In this context L is referred to as the set of *polyfaces* of $P_\alpha(L)$. The set of 2-dimensional polysolids in $\mathbb{R}^2$ is defined by

$$\mathcal{P}^2(\mathbb{R}^2) = \{P_\alpha(L) : L \in \mathcal{P}^1(\mathbb{R}^1) \text{ is finite and } \alpha \in \{0, 1\}\}. \tag{8}$$

The definition extends to planes embedded in spaces of dimension $d > 2$. If $H \subset \mathbb{R}^d$ is a 2-flat, then $\mathcal{P}^2(H)$ is defined to be the image of $\mathcal{P}^2(\mathbb{R}^2)$ under any affine transformation of $\mathbb{R}^2$ onto H. The set of all 2-dimensional polysolids in $\mathbb{R}^d$ is

$$\mathcal{P}^2(\mathbb{R}^d) = \bigcup \{\mathcal{P}^2(H) : H \text{ is a 2-flat in } \mathbb{R}^d\} \tag{9}$$

An important subset of $\mathcal{P}^2(\mathbb{R}^2)$ is the collection of pairs of bounded-cobounded polysolids, denoted $\mathcal{P}_b^2(\mathbb{R}^2)$. These polysolids are generated by sets L of bounded 1-dimensional polysolids. The set of bounded-cobounded polysolids corresponding to a 2-flat H in $\mathbb{R}^d$ is denoted $\mathcal{P}_b^2(H)$ and the set of 2-dimensional bounded-cobounded polysolids in $\mathbb{R}^d$ is denoted $\mathcal{P}_b^2(\mathbb{R}^d)$.

3.3 Polysolids in $\mathbb{R}^n$

Let $L \subset \mathcal{P}^{n-1}(\mathbb{R}^n)$ have a finite number of elements, each element an $(n-1)$-dimensional polysolid. The set

$$\mathbb{R}^n \setminus \mathrm{Closure}(L) = \mathbb{R}^n \setminus \bigcup_{e \in L} \mathrm{Closure}(e) \tag{10}$$

is an open set that is a disjoint union of a finite number of open connected sets, $V_L = \{U_i\}_{i=0}^m$.

The set L is said to be *decomposing in* $\mathbb{R}^n$ if and only if V_L is composed of regular open sets. A graph associated with L is $G_L = (V_L, E_L)$, where $(U_i, U_j) \in E_L$ if and only if $U_i \cup U_j$ is not a regular open set. The set L is said to be *generating in* $\mathbb{R}^n$ if and only if it is decomposing and G_L is 2-colorable.

A generating set L defines two complementary *polysolids*

$$P_\alpha(L) = \{U \in V_L : c(U) = \alpha\} \tag{11}$$

where $c : V_L \to \{0, 1\}$ is a 2-coloring of G_L. In this context the set L is referred to as the set of *polyfaces* of $P_\alpha(L)$.

The set of n-dimensional polysolids in $\mathbb{R}^n$ is defined by

$$\mathcal{P}^n(\mathbb{R}^n) = \{P_\alpha(L) : L \in \mathcal{P}^{n-1}(\mathbb{R}^{n-1}) \text{ is finite and } \alpha \in \{0, 1\}\} \tag{12}$$

The definition extends to hyperplanes in spaces of dimension $d > n$. If $H \subset \mathbb{R}^d$ is an n-flat, then $\mathcal{P}^n(H)$ is defined to be the image of $\mathcal{P}^n(\mathbb{R}^n)$ under any affine transformation of $\mathbb{R}^n$ onto H. The set of all n-dimensional polysolids in $\mathbb{R}^d$ is

$$\mathcal{P}^n(\mathbb{R}^d) = \bigcup\{\mathcal{P}^n(H) : H \text{ is an } n\text{-flat in } \mathbb{R}^d\} \tag{13}$$

An important subset of $\mathcal{P}^n(\mathbb{R}^n)$ is the collection of pairs of bounded-cobounded polysolids, denoted $\mathcal{P}^n_b(\mathbb{R}^n)$. These polysolids are generated by sets L of bounded $(n-1)$-dimensional polysolids. The set of bounded-cobounded polysolids corresponding to an n-flat H in $\mathbb{R}^d$ is denoted $\mathcal{P}^k_b(H)$ and the set of n-dimensional bounded-cobounded polysolids in $\mathbb{R}^d$ is denoted $\mathcal{P}^n_b(\mathbb{R}^d)$.

For general dimension n, a polysolid $\pi \in \mathcal{P}^n(\mathbb{R}^n)$ is described by its polyfaces $L = f(\pi)$ and the color of the polysolid $c(\pi)$. The implementation of Boolean operations on polysolids needs to determine both polyfaces and color.

4 Boolean Operations on Polysolids

Given polysolids in $\mathcal{P}^n(H)$, Boolean operations are defined on these using the topological nature of the polysolid definitions. Let $\pi = P_\alpha(L)$, $\pi_1 = P_{\alpha_1}(L_1)$ and $\pi_2 = P_{\alpha_2}(L_2)$ denote polysolids where L, L_1 and L_2 are generating sets and the polyfaces of the polysolids, $L = f(\pi)$, $L_1 = f(\pi_1)$ and $L_2 = f(\pi_2)$. The colors of the polysolids are $c_L = c(\pi)$, $c_{L_1}(\pi_1)$ and $c_{L_2}(\pi_2)$. Boolean operations are defined by

Negation:	$\neg\pi = P_{1-\alpha}(L)$
Intersection:	$\pi_1 \wedge \pi_2 = \pi_1 \cap \pi_2$
Union:	$\pi_1 \vee \pi_2 = \text{Interior}(\text{Closure}(\pi_1 \cup \pi_2))$
Difference:	$\pi_1 \neg \pi_2 = \pi_1 \cap \text{Complement}(\text{Closure}(\pi_2))$
Exclusive-Or:	$\pi_1 \oplus \pi_2 = (\pi_1 \neg \pi_2) \cup (\pi_2 \neg \pi_1)$.

The algorithm for computing a Boolean operation of two polysolids, say, $\pi = B(\pi_1, \pi_2)$, is briefly described next. Specific details for the 2D and 3D cases are provided later in the document.

The subsection on *normalization* involves decomposition of the polyfaces L_1 and L_2 into subsets whose elements are not intersecting. This is accomplished by a *segmentation* of each polyface of one polysolid relative to the other polysolid. The segmentation is recursive through dimension, reaching dimension 2 as the base case.. The essential work is done in segmenting a line containing an edge of one polysolid relative to the edges of a 2-dimensional polysolid.

The subsection on *acceptance* involves selecting normalized polyfaces from the input polysolids to combine into the polyfaces for the output. This is accomplished by maintaining *tags* for the segmented polyfaces of one polysolid relative to another polysolid.

4.1 Canonical Form for a Polysolid

The Maynard-Tavernini framework does not include the material of this section, but in practice it is useful to have a *canonical form* for the polysolids. The canonical form involves requirements for the color of the polysolids, for the features of the graph data structure representing the polysolid, requirements and for the directions of the normals to polyfaces.

4.1.1 Color Selection

Determination of the color $c(\pi)$ turns out to be trivial. The convention I use is that the bounded region of a polysolid has color 1 and the cobounded region of a polysolid has color 0. In this sense you can imagine rendering a 2D polysolid into a binary image of background pixels (0-valued) and foreground pixels (1-valued). The convention does not apply to the negation operator. The cobounded region of π is $P_0(L)$, the open sets having color 0. The bounded region of π is $P_1(L)$, the open sets having color 1. The cobounded region of the negation $\neg\pi$ is $P_1(L)$, the open sets having color 1. The bounded region of $\neg\pi$ is $P_0(L)$, the open sets having color 0. However, the negation operator is effectively used only for computing the Boolean difference and the Boolean exclusive-or, the output of these operations satisfying the convention.

4.1.2 Requirements for Graph Features

TODO: This section needs some figures to illustrate the ideas.

In 2D, a graph data structure for a polysolid will contain a set of vertices (positions) $\{\boldsymbol{V}_i\}$ and a set of edges $\{\langle \boldsymbol{V}_{i_0}, \boldsymbol{V}_{i_1}\rangle\}$. Depending on the graph implementation, the edges might be undirected or directed. The vertices and edges are referred to as the *graph features*.

A requirement is that the edges be directed so that as you traverse an edge, the bounded region adjacent to the edge is to your left. This is consistent with the definition of a simple polygon whose vertices are counterclockwise ordered. The edges of a simple polygon form a single closed polyline. On the other hand, polysolids can have multiple closed polylines. A simple polygon (outer polygon) with one ore more disjoint simple polygons (inner polygons) strictly contained in the outer polygon is such an example. These are usually referred to as polygons with holes. A bow-tie formed by two solid triangles that overlap only at a single vertex is another example. Let the vertices be $\boldsymbol{V}_0 = (0,0)$, $\boldsymbol{V}_1 = (2,0)$, $\boldsymbol{V}_2 = (1,1)$, $\boldsymbol{V}_3 = (0,2)$ and $\boldsymbol{V}_4 = (2,2)$. The directed edges are $E_0 = \langle \boldsymbol{V}_0, \boldsymbol{V}_1\rangle$, $E_1 = \langle \boldsymbol{V}_1, \boldsymbol{V}_2\rangle$, $E_2 = \langle \boldsymbol{V}_2, \boldsymbol{V}_3\rangle$, $E_3 = \langle \boldsymbol{V}_3, \boldsymbol{V}_4\rangle$ and $E_4 = \langle \boldsymbol{V}_4, \boldsymbol{V}_1\rangle$. As you traverse the directed edges, the bounded region is always to your left.

Another requirement is that the set of vertices are the only points at which edges can intersect. For example, consider the set of vertices $\boldsymbol{V}_0 = (0,0)$, $\boldsymbol{V}_1 = (2,0)$, $\boldsymbol{V}_2 = (0,2)$ and $\boldsymbol{V}_3 = (2,2)$ and the set of directed edges $E_0 = \langle \boldsymbol{V}_0, \boldsymbol{V}_1\rangle$, $E_1 = \langle \boldsymbol{V}_1, \boldsymbol{V}_2\rangle$, $E_2 = \langle \boldsymbol{V}_2, \boldsymbol{V}_3\rangle$ and $E_3 = \langle \boldsymbol{V}_3, \boldsymbol{V}_0\rangle$. This not a polysolid in canonical form because the edges E_1 and E_3 intersect at the point $(1,1)$, but this point is not listed in the set of vertices.

An example of a bounded polysolid representing a triangle with vertices $\boldsymbol{V}_0 = (0,0)$, $\boldsymbol{V}_1 = (2,0)$ and $\boldsymbol{V}_2 =$

$(0, 2)$ with edges $E_0 = \langle \boldsymbol{V}_0, \boldsymbol{V}_1 \rangle$, $E_1 = \langle \boldsymbol{V}_1, \boldsymbol{V}_2 \rangle$ and $E_2 = \langle \boldsymbol{V}_2, \boldsymbol{V}_0 \rangle$ has generating set $L = \{E_0, E_1, E_2\}$, where the edges are considered to be open in the sense that the endpoints are not included. Include an additional vertex $\boldsymbol{V}_3 = (0, 1)$ and choose the edges to be $E_0 = \langle \boldsymbol{V}_0, \boldsymbol{V}_1 \rangle$, $E_1 = \langle \boldsymbol{V}_1, \boldsymbol{V}_2 \rangle$, $E_2 = \langle \boldsymbol{V}_2, \boldsymbol{V}_3 \rangle$ and $E_3 = \langle \boldsymbol{V}_3, \boldsymbol{V}_0 \rangle$. The triangle has generating set $L' = \{E_0, E_1, E_2, E_3\}$. The line containing vertices $\boldsymbol{V}_2$, $\boldsymbol{V}_3$ and $\boldsymbol{V}_0$ is a 1-dimensional polysolid whose generating set has an even number of elements. As mentioned previously, the graph G_L is 2-colorable, but each colored component is unbounded. It is better to require that the generating set consist of an odd number of elements. But even this is perhaps still not desirable. If one were to add yet another vertex $\boldsymbol{V}_4 = (0, 1/2)$ and replace edges E_2 and E_3 by $E_2 = \langle \boldsymbol{V}_2, \boldsymbol{V}_3 \rangle$, $E_3 = \langle \boldsymbol{V}_3, \boldsymbol{V}_4 \rangle$ and $E_4 = \langle \boldsymbol{V}_4, \boldsymbol{V}_0 \rangle$, the line containing vertices $\boldsymbol{V}_2$, $\boldsymbol{V}_3$, $\boldsymbol{V}_4$ and $\boldsymbol{V}_0$ has a generating set with 4 elements, so the polyline has a bounded component and an unbounded component. Unfortunately, the interval corresponding to E_3 is a subset of the unbounded component. Although this does not cause problems in an implementation of Boolean operations, it is not desirable (as will become clear in 3D). Therefore, colinear vertices that are irrelevant to the topology of the 2D polysolid (the triangle) should be discarded from the graph data structure.

In 3D, a typical graph data structure for a polysolid will contain a set of vertices $\{\boldsymbol{V}_i\}$, a set of edges $\{\langle \boldsymbol{V}_{i_0}, \boldsymbol{V}_{i_1} \rangle\}$ and a set of triangles $\{\langle \boldsymbol{V}_{i_0}, \boldsymbol{V}_{i_1}, \boldsymbol{V}_{i_2}\}$. Usually the triangles are required to have ordered vertices as viewed by an observe close to the triangle, either counterclockwise or clockwise ordered. The edges are undirected because an edge shared by two triangles has different ordered vertices depending on the triangle used to visit it. For example, two consistently ordered triangles $\langle \boldsymbol{V}_0, \boldsymbol{V}_1, \boldsymbol{V}_2 \rangle$ and $\langle \boldsymbol{V}_3, \boldsymbol{V}_2, \boldsymbol{V}_1 \rangle$ share the edge $(\boldsymbol{V}_1, \boldsymbol{V}_2)$, but visiting the edge via the first triangle gives a directed edge $\langle \boldsymbol{V}_1, \boldsymbol{V}_2 \rangle$ and visiting the edge via the second triangle gives a directed edge $\langle \boldsymbol{V}_2, \boldsymbol{V}_1 \rangle$. It is possible to store both edge representations, but this comes at the cost of greater complexity in managing the graph data structure.

An additional complication using a vertex-edge-triangle graph is that adjacent and coplanar triangles can form a polygonal face. A simple example is a square face formed by the union of two triangles. The shared diagonal edge allows the two triangles to be decomposing regarding 2D polysolids, but not generating. A better choice to support for Boolean operations with polysolids is to represent the 3D polysolid with a graph data structure that has a set of 2D polysolids for the faces, each face represented by a set of 1D polysolids for the edges. In this sense, the vertices are the 0D polysolids that form the edges.

4.1.3 Requirements for Polyface Normals

4.2 Normalization of the Input Polysolids

4.3 Acceptance of Polyfaces for the Output Polysolid

1. Determine the color α of the result of the function. This step is trivial in the implementation.
2. Determine which subelements of L_1 and L_2 can be used to define the generating set L for the result of the function. This step requires the following.
   (a) Normalization. Decompose the polyfaces L_1 and L_2 into components which are non-intersecting.
   (b) Acceptance. Determine which of the normalized polyfaces to keep for the specified Boolean operation. This is accomplished by maintaining *tags* on the segmented polyfaces relative to a polysolid according to the following relationships.

o The polyface is *outside* the polysolid.

i The polyface is *inside* the polysolid.

+ The polyface is a *positive boundary* of the polysolid. That is, the polyface lies on the boundary of the polysolid with the interior of the polysolid to the positive side of the hyperplane in which the polyface lives.

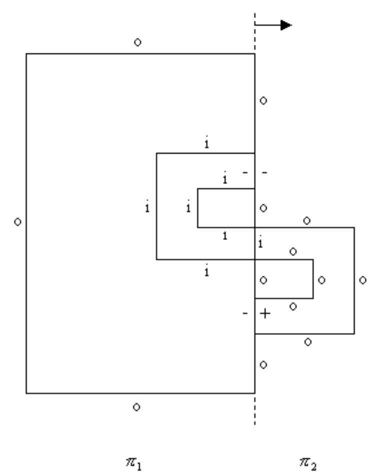
- The polyface is a *negative boundary* of the polysolid. That is, the polyface lies on the boundary of the polysolid with the interior of the polysolid to the negative side of the hyperplane in which the polyface lives.

The tags form the Klein-4 group whose binary operation is defined in the table below.

(t_0, t_1)	o	i	-	+
o	o	i	-	+
i	i	o	+	-
-	-	+	o	i
+	+	-	i	o

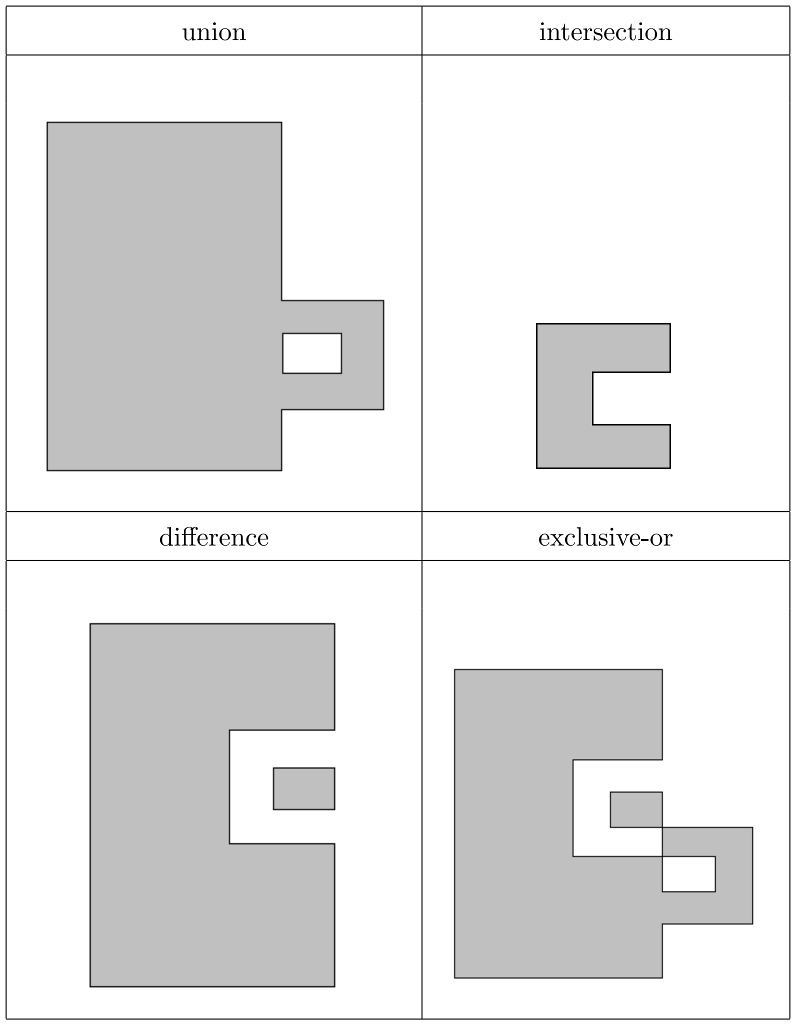
The group operation of two elements t_0 (select row) and t_1 (select column) is denoted $t_2 = (t_0, t_1)$; for example, $i = (-, +)$. The algorithm involves building four lists of polyfaces, one list per tag. The lists are merged according to the Boolean operation, the final list yielding the resulting polysolid.

Figure 3. Segmentation of two bounded polysolids.



To briefly illustrate, consider two polysolids, π_1 a square and the π_2 an s-shaped object. They are shown normalized and superimposed in Figure 3 with the various tags on the edges. The union, intersection, difference, and exclusive-or are shown in Figure 4.

Figure 4. Boolean operations of the two polysolids in Figure 3. Union of the two polysolids.



In the notes from Maynard and Tavernini, the line shown is drawn as an oriented line with normal pointing to the right. That oriented line was used for segmentation for both polysolids. In my implementation, the line normal is chosen to point to the side containing the bounded part of the segmenting polysolid. As such, my tags differ from theirs on a subedge which has the bounded parts of the two polysolids to opposite sides of that subedge There is one such subedge in Figure 3. The Maynard algorithm will mark the subedge with a $+$ and a $-$. My algorithm marks it with a $+$ and a $+$. When comparing two subedges for acceptance into a union, the Maynard algorithm sees that the subedge tags multiply to i and is rejected. In my algorithm, the subedge tags multiply to o, but the edge is still rejected because I compare the two subedges assuming that the end points are ordered. That is, the line has a specific direction vector determined from its normal vector and the subedges are constructed in the segmentation using the line direction to order the end points. While the subedges have the same tags, they have opposite directions. In the Maynard algorithm, the subedges have different tags and the same direction.

5 Boolean Operations on Polysolids in $\mathbb{R}^2$

5.1 Normalization

Each polysolid is segmented against the edges of the other polysolid. If P and Q are the two polysolids, segmenting Q against P is given by the pseudocode of Listing 1.

Listing 1. Pseudocode for segmenting one polysolid against the edges of another polysolid.

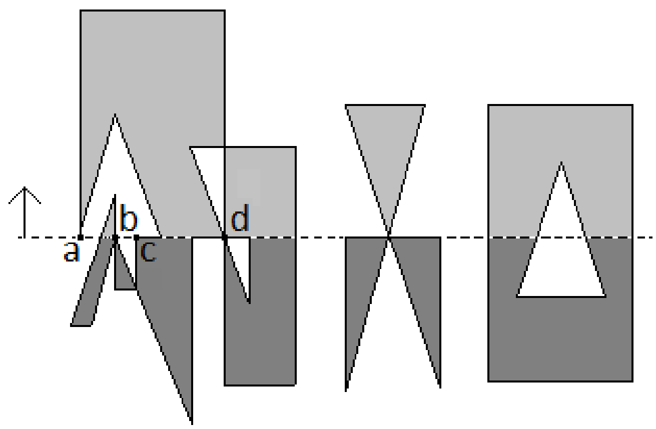
```
for each edge E of Q do
{
    L = directed line containing E;
    S = segmentation of L by P;
    if S is not empty then
    {
        // L intersects P
        S = S intersected with E;
        if S is not empty then
        {
            compute tagged-edge lists T[o], T[i], T[+] and T[i] from S;
        }
    }
    else // S is empty
    {
        // L does not intersect P, E is outside
        add E to tagged-edge list T[o];
    }
}
```

The tagged edge lists are used in the acceptance phase, the topic of the next subsection.

The key operation is the segmentation of a line L by a polysolid P. This is done by iterating over all edges of P and determining if and edge E amd L (1) intersect at an interior point of E, (2) intersect at an end point of E, or (3) do not intersect. In case (1) the tag on the intersection is i. In case (2) the tag on the intersection is $+$ if E is on the positive side of the line (the side to which the normal vector points) or i if E is on the negative side of the line. If E is contained in L, then no tagging is necessary (edges of P parallel to L need not be processed).

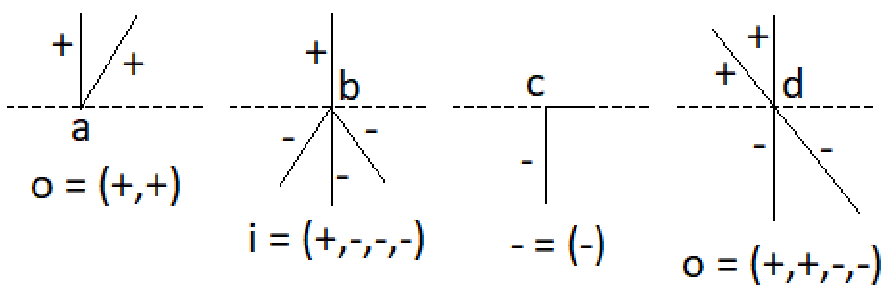
Figure 5 shows a line and a polysolid consisting of three components, the last of which has a hole. The normal vector for the line is drawn and the positive side of the line is that side pointed to by the normal. The tagging of the intersection points is described for the four points labeled a, b, c, and d. The shading is used to help visual how the polysolid is built from the edges and which parts are above or below the line.

Figure 5. Polysolid segmenting a line.



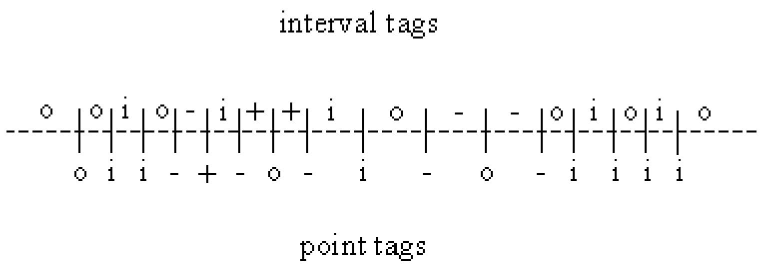
At point a there are two edges intersecting the line. The common intersection point has tag + from the first edge and tag + from the second edge. The final tag on the intersection point is the Klein-4 product of the tags, $o = + \cdot +$. At point b there are three edges intersecting the line. The tags are $-$, $-$, and i. The final tag is the product $i = - \cdot - \cdot i$. At point c there are two edges, but one edge is contained in the line and can be ignored. The final tag is $-$, the tag generated from the other edge. At point d there are two edges intersecting the line, both with a i tag. The final tag is the product $o = i \cdot i$. Figure 6 shows the four situations.

Figure 6. Computation of point tags.



For the entire line there are 16 points of intersection (counting only the two end points for each edge contained in the line). The intervals are tagged starting with the left-most half-infinite interval having tag o. Traversing from left to right, the tag on the next interval is the tag of the previous interval times the tag of the point separating the two intervals. Figure 7 shows the point and interval tags.

Figure 7. Point and interval tags.



This shows how the Klein-4 group multiplications allow the tagging of higher dimensional structures from the lower dimensional ones. This idea carries over into polysolids in higher dimensions.

5.2 Acceptance

After normalization we have four lists of tagged edges corresponding to the o, i, $+$, and $-$ tags for each of the two polysolids. These lists are merged as described below to obtain the various Boolean combinations of the polysolids. It is instructive to apply these rules to the polysolids in Figure 1 to obtain the Boolean results in Figure 2.

- UNION. All o tagged edges are in the union. Pairs of edges having the same direction and both $+$ tags or both $-$ tags are in the union.
- INTERSECTION. All i tagged edges are in the intersection. All $+$ tags are in the intersection (duplicates between the two $+$ lists must be avoided).
- DIFFERENCE. Let the polysolids be labeled P and Q. The difference is $P \setminus Q$. This can be thought of as the intersection of P and $\neg Q$ where the negation indicates to change the color of the polysolid. When comparing P against Q, the tags on P are computed based on the bounded portion of Q. To compare against $\neg Q$ requires the tags on P to be negated (in a sense). Thus, all o tagged edges of P and all i tagged edges of Q are in the difference. When merging, the directions of the i tagged edges of Q must be reversed. Also, all $+$ tagged edges are kept in an intersection, but because we are comparing against $\neg Q$, the $-$ tagged edges of P and the $+$ tagged edges of Q are in the difference (duplicates between these two lists must be avoided).
- EXCLUSIVE OR. For polysolids P and Q, this is simply computed as the union of the differences $P \setminus Q$ and $Q \setminus P$.